Power of Two choices [Azar-Broder-Karlin-Upfal]

- n bins ; n balls ; for each ball, choose two bins at random and throw ball into less loaded bin.

Analysis

Fact 1: By Chernoff bound, $\Pr[\sum X_t > e\mu] \le \left(\frac{e^{e-1}}{(1+e-1)^{1+e-1}}\right)^{\mu} = e^{-\mu}$.

$\varepsilon = e-1$

$\beta_6 = \frac{n}{2e}$; $\beta_{i+1} = \frac{e\beta_i^2}{n}$.

ν_i = # of bins with $\geq i$ balls

μ_i = # of balls which were placed in bins already containing $\geq i-1$ balls

Fact 2: $\mu_i \geq \nu_i$

Fix i. $X_t = 1$ if t^{th} ball contributes to μ_{i+1}.

$\mathcal{E}_i$: Event $\nu_i \le \beta_i$

Lemma: If $\beta_i^2 \geq 2n\log n$, then $\Pr(\mathcal{E}_i) \geq 1 - \frac{1}{n}$.

Proof: $\Pr[\neg\mathcal{E}_{i+1} | \mathcal{E}_i] \le \Pr[\mu_{i+1} > \beta_{i+1} | \mathcal{E}_i]$ (by Fact 2)

$= \Pr(\sum_t X_t > \beta_{i+1} | \mathcal{E}_i]$

$\le e^{-\left(\frac{\beta_i}{n}\right)^2 n}$ (by Fact 1)

$= e^{-\beta_i^2/n}$

Since $\beta_i^2 \geq 2n\log n$, $\Pr(\neg\mathcal{E}_{i+1} | \mathcal{E}_i) \le 1/n^2$.

$\Pr(\neg\mathcal{E}_{i+1}) \le \Pr(\neg\mathcal{E}_{i+1} | \mathcal{E}_i] + \Pr(\neg\mathcal{E}_i] \le \ldots \le \sum \Pr[\neg\mathcal{E}_{j+1} | \mathcal{E}_j]$

$\le 1/n$.

Lemma: If $\beta_i^2 \le 2n\log n$, $\Pr[\nu_{i+2} \geq 1] < 1/n$.

Proof: $E(\mu_{i+1} | \mathcal{E}_i) \le n\cdot\left(\frac{\beta_i}{n}\right)^2 \le 2\log n$

$\Rightarrow \Pr(\mu_{i+1} > 2e\log n | \mathcal{E}_i) \le e^{-2\log n} = 1/n^2$

$\Rightarrow \Pr(\nu_{i+1} > 2e\log n \mid \mathcal{E}_i) \leq 1/n^2$ (by Fact 1)

Let $\mathcal{E}_{i+1}$ denote $\nu_{i+1} \leq 2e \log n$.

$$\Pr(\nu_{i+2} \geq 1 \mid \mathcal{E}_{i+1}) = \Pr(\mu_{i+2} \geq 1 \mid \mathcal{E}_{i+1})$$

$$\leq n \cdot \left(\frac{2e\log n}{n}\right)^2 = o(1) \quad \text{(by union bound)}$$

Theorem: max load $\leq O(\log\log n)$ whp

Proof: $\beta_i^2 = \left(e^{i-6} \cdot \frac{n}{(2e)^{2^i}}\right)^2 = 2n\log n$

$$\Rightarrow i = \theta(\log\log n)$$