Integrality gap = $\max_{\text{instances}} \left\{ \dfrac{\text{integral opt}}{\text{fractional opt}} \right\}$ (for minimization)

Set cover LP

$$\min \sum c_s x_s$$

$$\sum_{s : e \in s} x_s \geq 1$$

$$x_s \geq 0$$

U = Binary strings of length k

$\mathcal{S} = \{ S_e : e \in U \}$ with $c_{S_e} = 1 \;\forall\, S_e$.

where $S_e = \{ e' \in U : \#$ of 1's in $e \wedge e' = $ odd$\}$

Clearly $|S_e| = 2^{k-1}$

Also, each element is in 2^{k-1} sets.

$\Rightarrow x_s = \frac{1}{2^{k-1}}$ is feasible and has cost 2

Lemma: Any integer solution must contain at least k sets.

Proof. If sets $S_{e_1}, S_{e_2}, \ldots, S_{e_p}$ are chosen, then $e_1, e_2, \ldots, e_p$ must span the hypercube. Otherwise, $\exists\, e \in U$ s.t. e is orthogonal to each of $e_1, e_2, \ldots, e_p$ and therefore $e \notin S_{e_1}, S_{e_2}, \ldots, S_{e_p}$.

Thus, $p \geq k$.