Metric Facility Location (Jain-Vazirani)

- Facilities j with facility cost c_j
- Clients i with distance d_{ij} to facility j

$$\min \sum_j c_j x_j + \sum_i \sum_j d_{ij} y_{ij} \qquad \Big| \qquad \max \sum_i \alpha_i$$

$$x_j \geq y_{ij} \quad \to \beta_{ij} \qquad \Big| \qquad \alpha_i - \beta_{ij} \leq d_{ij}$$

$$\sum_j y_{ij} \geq 1 \quad \to \alpha_i \qquad \Big| \qquad \sum_i \beta_{ij} \leq c_j$$

How to interpret this dual? Observe that wlog we can assume

$$\beta_{ij} = \max(\alpha_i - d_{ij}, 0)$$

So, in the dual program, if we grow α_i, then it can grow till d_{ij} (for any j), at which point β_{ij} must also grow with α_i.

<u>Algo</u>: Grow α_i uniformly; when $\alpha_i \geq d_{ij}$, grow β_{ij} as well. When a facility becomes tight, buy it in the primal and stop growing all clients that are tight (i.e. $\alpha_i \geq d_{ij}$).

Problem: α_i contributes to multiple β_{ij}.

<u>Fix</u>: Phase 2: Construct conflict graph on facilities where (i, i') are conflicted if $\exists j'$ s.t. j' is tight for both i and i'. Buy a maximal independent set of conflict graph in the same order as phase 1. Connect each client either to its mapped facility if it is in the MIS

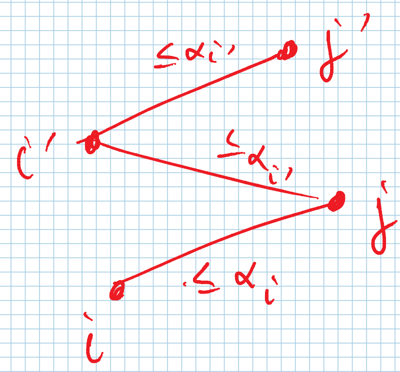
or to the neighbor in the conflict graph of its mapped facility.

__Claim__: The dual can pay for all facility costs and connection cost on first edge of connection path for all clients.

__Proof__: Since no two facilities are congested by the same α_i, the proof is immediate.

__Claim__: The connection cost on each of the last two edges of a client i is at most α_i each.

__Proof__:



$\alpha_{i'} \le \alpha_i$ since j' became congested before j.