COMPSCI 330: Design and Analysis of Algorithms 11/13/2014

Lecture #22

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1 Overview

In this lecture we reviewed the general definition of linear programming, as well as the properties of primal and dual linear programs. We then introduced two linear programming duality theorems, and analysed their application on Maximum Flow.

2 Review of Linear Programming

2.1 Primal and Dual Linear Program

Definition 1. *A* ***primal linear program*** *is in the form of,*

$$\begin{aligned} \textit{maximize} \quad & \mathbf{c^T x} \\ \textit{subject to} \quad & \mathbf{Ax} \leq \mathbf{b} \\ & \mathbf{x} \geq \vec{0} \end{aligned}$$

Definition 2. *The corresponding* ***dual linear program*** *is in the form of,*

$$\begin{aligned} \textit{minimize} \quad & \mathbf{b^T y} \\ \textit{subject to} \quad & \mathbf{A^T y} \geq \mathbf{c} \\ & \mathbf{y} \geq \vec{0} \end{aligned}$$

2.2 Dual of Dual Linear Program

Imagine that we have the following **primal linear program**:

$$\begin{aligned} \text{maximize} \quad & \mathbf{x} + \mathbf{y} \\ \text{subject to} \quad & \mathbf{a}(\mathbf{x} + 2\mathbf{y} \leq 4) \\ & \mathbf{b}(2\mathbf{x} + \mathbf{y} \leq 8) \\ & \mathbf{x} \geq \vec{0} \\ & \mathbf{y} \geq \vec{0} \end{aligned}$$

To find out the **a** and **b** that maximise the result we compute the corresponding **dual linear program**:

$$\begin{aligned} \text{minimize} \quad & 4\mathbf{a} + 8\mathbf{b} \\ \text{subject to} \quad & \mathbf{a} + 2\mathbf{b} \geq 1 \\ & 2\mathbf{a} + \mathbf{b} \geq 1 \\ & \mathbf{a} \geq \vec{0} \\ & \mathbf{b} \geq \vec{0} \end{aligned}$$

To compute the **dual of dual linear program** we have:

$$\begin{aligned} \text{minimize} \quad & 4\mathbf{a} + 8\mathbf{b} \\ \text{subject to} \quad & \mathbf{c}(\mathbf{a} + 2\mathbf{b} \geq 1) \\ & \mathbf{d}(2\mathbf{a} + \mathbf{b} \geq 1) \\ & \mathbf{a} \geq \vec{0} \\ & \mathbf{b} \geq \vec{0} \end{aligned}$$

Which gives:

$$\begin{aligned} \text{maximize} \quad & \mathbf{c} + \mathbf{d} \\ \text{subject to} \quad & \mathbf{c} + 2\mathbf{d} \leq 4 \\ & 2\mathbf{c} + \mathbf{d} \leq 8 \\ & \mathbf{c} \geq \vec{0} \\ & \mathbf{d} \geq \vec{0} \end{aligned}$$

Thus, we know that,

Definition 3. *The **dual of dual linear program** is exactly the **primal linear program**.*

3 Value of Linear Programs

3.1 Weak Linear Program Duality Theorem

Theorem 1. *Let x^*, y^* be any feasible solutions of a primal-dual pair for,*

$$\begin{aligned} \textit{maximize} \quad & \mathbf{c}^{\mathbf{T}}\mathbf{x} \\ \textit{subject to} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \vec{0} \end{aligned}$$

$$\begin{aligned} \textit{minimize} \quad & \mathbf{b^T y} \\ \textit{subject to} \quad & \mathbf{A^T y} \geq \mathbf{c} \\ & \mathbf{y} \geq \overrightarrow{0} \end{aligned}$$

Then,

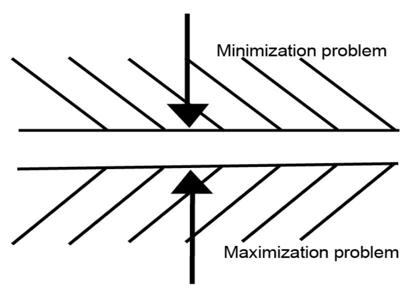
$$\mathbf{c^T x}* \leq \mathbf{b^T y}*$$

Proof.

$$\begin{aligned} \mathbf{c^T x^*} &= \mathbf{c^T A}^{-1}(\mathbf{Ax}^*) && <= \mathbf{A}^{-1}\mathbf{A} = 1 \\ &\leq \mathbf{c^T A}^{-1}\mathbf{b} && <= \text{rule from primal LP} \\ &\leq (\mathbf{A^T y})^{\mathbf{T}}\mathbf{A}^{-1}\mathbf{b} && <= \text{rule from dual LP} \\ &= \mathbf{y}^{*\mathbf{T}}\mathbf{AA^{-1}b} \\ &= \mathbf{y}^{*\mathbf{T}}\mathbf{b} \\ &= (\mathbf{b^T y}^*)^{\mathbf{T}} \\ &= \mathbf{b^T y}^* && <= \textit{as } \mathbf{b} \textit{ and } \mathbf{y}^* \textit{ are only scalars} \end{aligned}$$

□

Remark 1. *Feasible solutions are solutions that satisfy all the constraints. The weak linear program duality theorem shows that for all feasible solutions, the objective of the maximisation problem cannot exceed the objective of the minimisation problem. In other words, the two objectives bound each other.*



3.2 Strong Linear Program Duality Theorem

Theorem 2. *If in the above case,* $\mathbf{x}^*$, $\mathbf{y}^*$ *are also* ***optimal*** *for the primal-dual pair, then,*

$$\mathbf{c^T x}* = \mathbf{b^T y}*$$

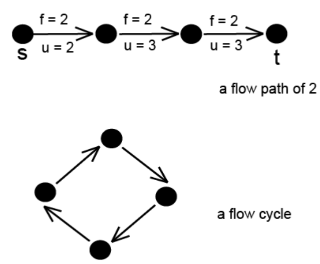
(not proven in class)

4 Application on Weak & Strong Duality Theorems - Maximum Flow

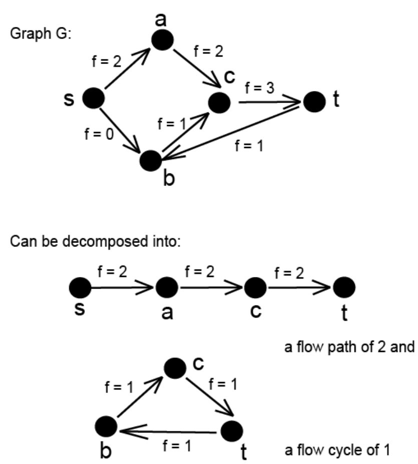
4.1 Flow Decomposition Theorem

Theorem 3. *Any flow can be decomposed into at most m flow paths and flow cycles.*

Remark 2. *A **flow path** is a path carrying a flow. A **flow cycle** is a cycle carrying a flow.*



For example, the below graph can be decomposed into a flow path of 2 and a flow cycle of 1:



Claim 4. *There always exists a maximum flow where the flow decomposition has no flow cycle.*

Proof. The value of the maximum flow is simply the sum of all flow paths, as for any flow cycle, whatever flow that goes into t will go out of t, which does not add any value into the total value of the flow. Thus, we could simply remove the cycles from a decomposition to get a new flow with the same maximum flow value. □

4.2 Linear Programming Representation

The **primal of maximum flow** can be represented as:

maximize Σf_p

subject to $p \in p(s,t)$ $<= p(s,t)$: the set of all paths from s to t

$\Sigma_{p:e\in p} f_p \leq u_e \; \forall e \in E$ $<=$ capacity constraint

$<= p : e \in p$: for all paths that contain the edge

$f_p \geq 0$

(The flow balance is automatically satisfied, as we are looking at flow paths.)

The **dual of maximum flow** can be represented as:

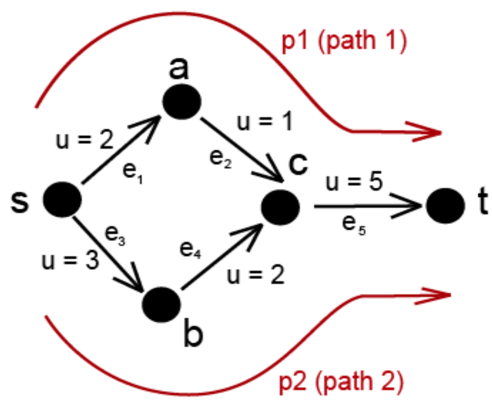
(we denote $\mathbf{x_e}$ as the dual variable)

minimize $\Sigma_{e\in E} u_e x_e$ $<=$ inferred from constrains of primal

subject to $\Sigma_{e\in p} x_e \geq 1 \; \forall p \in p(s,t)$

$x_e \geq 0$

For example, for the below graph, we have the **primal** linear program:



maximize $f_{p_1} + f_{p_2}$

subject to $f_{p_1} \leq 2$

$f_{p_1} \leq 1$

$f_{p_2} \leq 3$

$f_{p_2} \leq 2$

$f_{p_1} + f_{p_2} \leq 5$ $<=$ for e_5

To compute the **dual** linear program we have:

$$\begin{aligned} \text{maximize} \quad & f_{p_1} + f_{p_2} \\ \text{subject to} \quad & x_{e1}(f_{p_1} \leq 2) \\ & x_{e2}(f_{p_1} \leq 1) \\ & x_{e3}(f_{p_2} \leq 3) \\ & x_{e4}(f_{p_2} \leq 2) \\ & x_{e5}(f_{p_1} + f_{p_2} \leq 5) \end{aligned}$$

Which gives:

$$\begin{aligned} \text{miminize} \quad & 2x_{e1} + x_{e2} + 3x_{e3} + 2x_{e4} + 5x_{e5} && <= \text{this is essentially } \Sigma_{e \in E} u_e x_e \\ \text{subject to} \quad & x_{e1} + x_{e2} + x_{e5} \geq 1 \\ & x_{e3} + x_{e4} + x_{e5} \geq 1 && <= \text{this is essentially } \Sigma_{e \in p} x_e \geq 1 \; \forall p \in p(s,t) \end{aligned}$$

Remark 3. *The **dual linear program of maximum flow** is essentially **fractional minimum cut** problem. (it* is *minimum cut if x_e is either* 0 *or* 1*).*

Remark 4. *The **primal linear program of maximum flow** has exponential number of variables, as #paths is exponential, and only polynomial number of constraints, as #constraints is essentially #edges. Thus, even though we could not solve the primal in polynomial time, we could use the **Strong Linear Program Duality Theorem** to convert the primal to the dual, solve the dual in polynomial time, and know that the solution (the value of the maximum flow) must also be the optimal solution for the primal.*

4.3 Equality of Minimum Cut and Fractional Minimum Cut

So far we've shown that:

$$\begin{aligned} \text{maximum flow} \quad & = \text{fractional minimum cut} && (1) \\ & \leq \text{minimum cut} && (2) \end{aligned}$$

(1) Based on the strong duality theorem. Fractional mincut is the optimal solution for the dual.
(2) Minimum cut has more restriction in the values x_e could take. It is always a feasible but not necessarily an optimal solution for the dual.

From the previous lecture we've learnt that fractional minimum cut *is* in fact equal to minimum cut. Here we prove the equality with our linear programs.

Theorem 5. *Fractional minimum cut = minimum cut.*

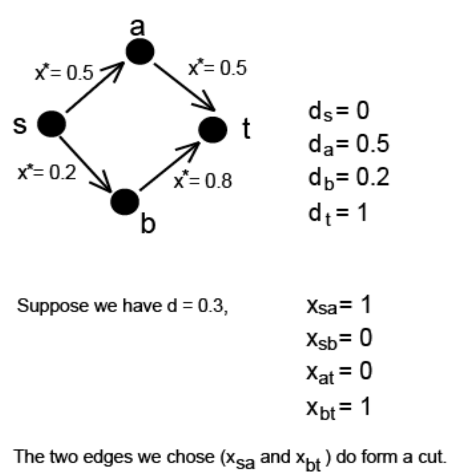
Proof. Suppose x_e^*, $e \in E$ is an optimal solution for the fractional minimum cut.

Define d_t as the length of the shortest path from s to t under x_e^*. We know that $d_t \geq 1$ because of the constrain from the dual linear program.

Suppose there is an edge from u to v. We know that $d_v \leq d_u + x_{uv}^*$.

Define a **random** cut $(S, \overline{S})$ as follows,

1. choose $d \in [0, 1]$ uniformly at random.
2. choose $x_{uv} = 1$ $if\ du \leq d \leq dv$ and 0 $otherwise$.



Claim 6. *The above randomised procedure will always form a cut. (an example shown above)*

Proof. We prove this claim by contradiction. Suppose the above procedure did not result in a cut. Then there must be some path that we haven't chosen any edge from. However, this is not possible as for any path we have $d_s = 0$ and $d_t = 1$. Thus, there muse be at least one edge that crosses d. □

We've shown that the above procedure will always form some cut, though the value is not deterministic since the cut is based on a randomised variable d. Now we want to show that the expected value of this cut is indeed the fractional minimum cut value that we start with.

Claim 7. $E[\Sigma_{e \in E} u_e x_e] \leq \Sigma_{e \in E} u_e x_e^*$

Proof.

$$E[\Sigma_{e \in E} u_e x_e] = \Sigma_{e \in E} u_e E[x_e] \quad (3)$$

$$\leq \Sigma_{e \in E} u_e x_e^* \quad (4)$$

(4): $E[x_e] = Pr[x_e = 1] = Pr[d_u \leq d \leq d_v] = d_v - d_u \leq x_{uv}^*$ (shown above) where $e = (u, v)$

□

The above claim indicates that there must exist a cut $(S,\overline{S})$ $s.t.$,

$$\Sigma_{e \in (S,\bar{S})} u_e \leq \Sigma_{e \in E} u_e x_e^* \text{ (the fractional minimum cut)}$$

as not all values can be greater than the expectation ($\Sigma_{e \in E} u_e x_e^*$ in this case).

We've shown previously that,

$$\text{fractional minimum cut} \leq \text{minimum cut}$$

Thus, the fractional minimum cut must be equal to the minimum cut for both equations to be valid. $\square$