Introduction to Approximation Algorithms

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CPS 570

Covered Today

- Approximation in general
- Set cover
- A greedy algorithm for set cover
- Submodularity
- A generic, greedy algorithm exploiting submodularity

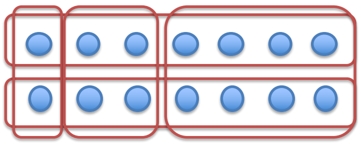
Why use approximation?

- Lots of problems we want to solve are NP-hard optimization problems, often with associated NP-complete decision problems
- Different notions of approximation
  - Search for a "pretty good" answer
  - Return an optimal answer in some cases (fail in others?)
  - Return an answer that is an additive factor from optimal: result = optimal +/- ε
  - Return an answer that a multiplicative factor from optimal: result/approximation = ε
  - For a given resource level, achieve a lower performance value?
  - For a given performance level, consume more resources?

Set Cover

- Input:
  - A set of atoms: $S=s_1 \ldots s_n$
  - A set of sets: $C=c_1 \ldots c_m$
  - Each set contains 1 or more atoms
- Optimization question: Can you choose k elements from C such that every element of S is in at least one of these k? (This is a called a cover.)
- Decision question: Exist a cover of size k or less?
- NP-hard

Set Cover Example



14 atoms
5 sets

Real Problems Abstracted by Set Cover

- Sensor placement:
  - You have sensors to place in m different locations
  - Each location can observe some fraction of your n targets
  - Find the most efficient sensor allocation to see all targets
- Buying bundles of goods
  - Different vendors offer package deals on different combinations of products (flat rate shipping)
  - Buy all the products you need in the smallest number of transactions
- Choosing advertising outlets
  - Different stations (or newspapers) cover different, possibly overlapping markets
  - Try to cover markets with smallest number of ads

So, what do we do?

- Settle for a larger k?
  - What if we don't need the absolute smallest k?
  - Is there an algorithm that gives something close to the smallest?
- Settle for less than full coverage
  - What if we have only k resources?
  - Is there an algorithm that gives us something close to the best we can hope for using k?

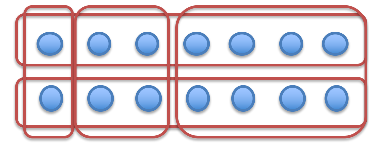
Greedy Algorithms

- Greedy algorithms are a general class of algorithms that, loosely speaking, make a choice that gives maximal short term improvement, without considering subsequent choices
- Examples of greedy behavior:
  - Picking the class that is most interesting to you first (ignoring that this might cause scheduling problems with other classes)
  - Positioning a sensor so that it sees the highest number of targets (while ignoring subsequent choices)

Greedy Set Cover

- Repeat until done*
  - For each set not added, check how many previously uncovered atoms it would add
  - Add the set with the biggest increase in the number of atoms covered
- *What is "done"
  - Max of k elements added, ***or***
  - All elements covered

What does greedy do here?



What price greed?

- Assume we have a budget of k
- Optimal picks: $O_1 \ldots O_k$, covering n atoms
- Greedy picks $G_1 \ldots G_k$, covering x atoms
- What is the relationship between x and n?

What price greed (2)?

- o_i = number of *new* elements covered by O_i
- g_i = number of *new* elements covered by G_i
- $n = o_1+o_2+\ldots+o_k$
- $x = g_1+g_2+\ldots+g_k$

What price greed (3)?

- Suppose $o_i > g_i$
- Q: Why didn't greedy pick O_i?
- A: The only reason would be if greedy already covered $o_i - g_i$ of the elements in O_i in some G_j, $j<i$
- $x \geq (o_1-g_1)+(o_2-g_2)+\ldots+(o_k-g_k)=n-x$
- $2x \geq n$
- $x \geq n/2$
- Conclusion: For fixed k, greedy gets a least half as much coverage as optimal

What about minimizing k?

- Suppose optimal coverage uses k to cover n atoms
- Assume we run greedy until it covers everything, taking h>k resources
- Analyze greedy's h choices in batches of k
  - Greedy covers at least n/2 in first batch of k
  - Second batch of k covers at least half of remaining atoms. Why? Same analysis can be repeated.
- Conclusion: greedy requires at most $k*\log_2 n$ resources
- Note: Our bounds here are not tight. Better proof exploiting submodularity is possible.

Applying to Other Problems

- If we have a good approximation scheme for one NP-complete problem, does this imply a good approximation scheme for others?
- Depends upon what what you mean by *"good"*…
- The polynomial factor can be a killer here
- Conclusion: Approximation algorithms will tend to be problem specific unless one discovers a more general approach to approximation

Submodularity

- f is a function defined on sets
- Submodular if:

$$X,Y \subseteq \Omega, X \subseteq Y : f(X \cup \{z\}) - f(X) \geq f(Y \cup \{z\}) - f(Y)$$

- Monotone if

$$X \subseteq Y : f(Y) \geq f(X)$$

Submodularity in English

- Adding to a subset has more "bang" than adding to a superset, or
- Diminishing returns for adding to bigger sets

- Monotonicity in English: Bigger is better (though not strictly)

Set Cover?

- Does set cover fit this framework?
- f = number of atoms covered
- Set Ω=C

- Is it submodular?
- Is it monotone?

Maximizing Monotone Submodular Set Functions

- This is NP-hard in general ☹
- Greedy algorithm for maximizing monotone submodular set functions is a 1-1/e factor from optimal
- Can use similar argument to set cover to get a resource bound
- Proof in reading, similar to our 2X bound, but a little more subtle

- This provides a generic procedure for analyzing greedy algorithms for certain classes of hard problems ☺

Greedy Set Cover and Submodularity

- Our greedy algorithm for set cover can be understood as an instance of the greedy approach for submodular set functions

- Conclusion: We get a tighter bound for free!
- (1-1/e > ½)

Conclusions

- Avoid worst consequences NP-hardness with clever approximation algorithms (or clever analysis of simple algorithms)
- Caveats:
  - Not all problems admit good approximate solutions
  - Approximation techniques for particular problems don't always carry over to others
- Some generic approaches exist:
  - Greedy algorithms sometimes do well
  - Submodularity provides a generic framework for analyzing certain types of greedy algorithms
  - Other families of approaches exist as well – rounding, LP relaxations, etc.