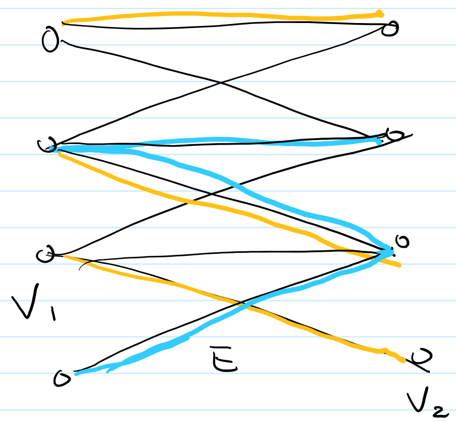
Lecture 16 Bipartite Matching

Tuesday, October 31, 2017 2:46 PM

- bipartite graph
  - A bipartive graph $G = (V_1, V_2, E)$, E is a subset of (i,j) $i \in V_1, j \in V_2$.

    (V_1: courses V_2: classrooms $(i,j) \in E$: course #i can be assigned to classroom #j)

  - A matching M is a subset of E, such that edges in M do not share vertices.



$M: \{(1,1), (2,3), (3,4)\}$

$|M| = 3$

blue: augmenting path

The size of a matching M is just the # of edges in M.

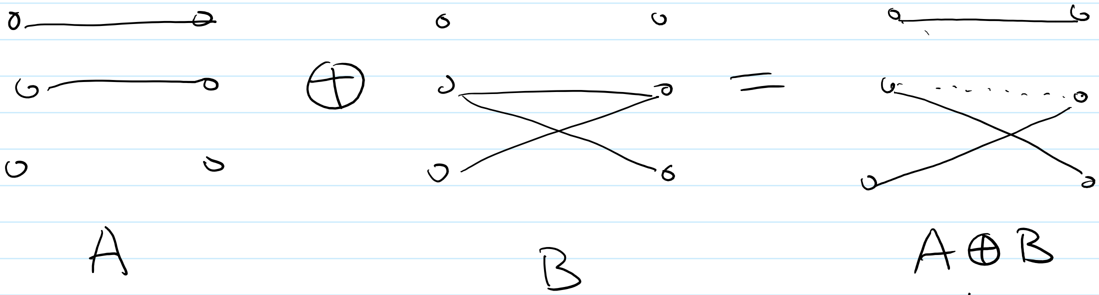
- Given a bipartite graph G and matching M.
  - an edge e is matched if $e \in M$, unmatched if $e \notin M$
  - a vertex is matched if it's connected to some $e \in M$, unmatched otherwise.
  - augmenting path P is a path from an unmatched vertex in V_1 to an unmatched vertex on V_2, and the edges alternate between unmatched and matched.

Claim: An augmenting path P has an odd # of edges, and it has exactly 1 more unmatched edges than matched edges.

- Xor operation: If A, B are two subsets of edges,

$A \oplus B$ is also a subset of edges

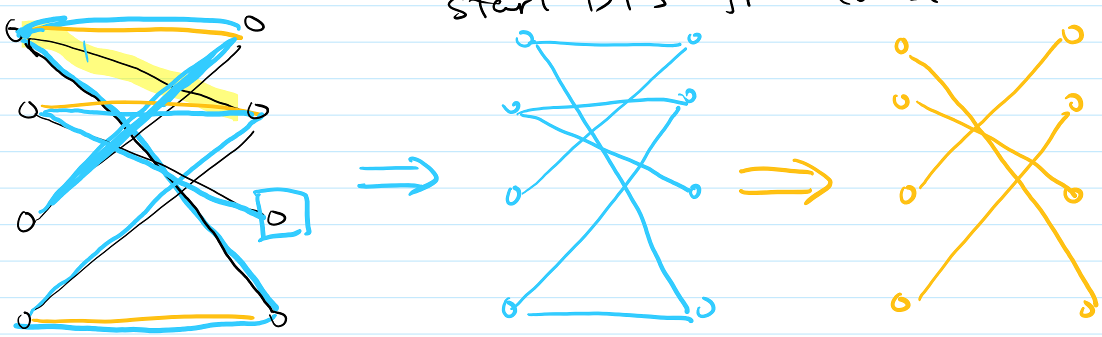
$$e \in A \oplus B \text{ if } \begin{cases} e \in A, \ e \notin B \\ e \notin A, \ e \in B \end{cases}$$



Claim: if P is an augmenting path for M, then $M' = M \oplus P$ is also a matching, and $|M'| = |M| + 1$

- Example of DFS for augmenting path

start DFS from course #3



- Thm for correctness:

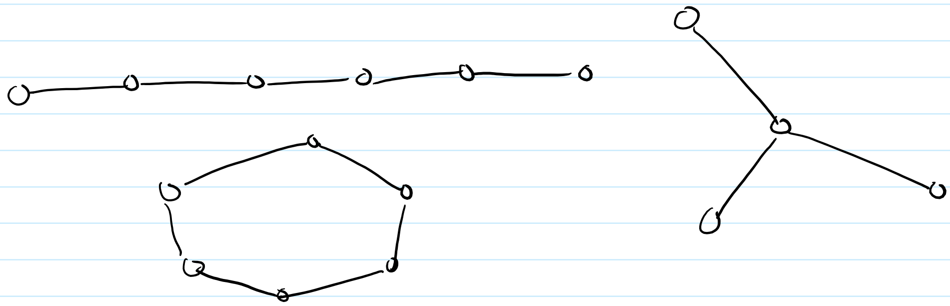
Proof by contradiction:

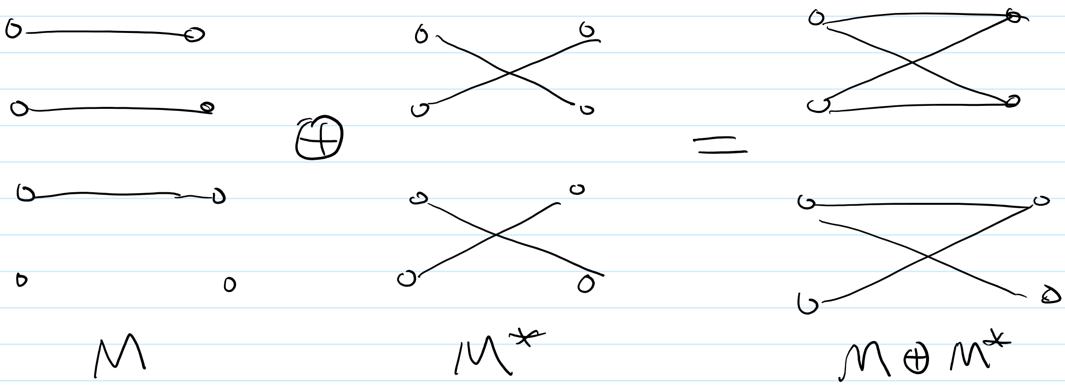
assume there is a larger matching M^* $(|M^*| > |M|)$

$\Longrightarrow$ there is an augmenting path.

Proof: Look at $M \oplus M^*$

$M \oplus M^*$: $\begin{cases} \text{each vertex is connected to } \leq 2 \text{ edges} \\ \text{union of vertex-disjoint paths and cycles} \end{cases}$





- in a cycle: (1) same # of edges in M, M^*

in a path either (2) $\boxed{M^* \text{ has 1 more edge}}$ augmenting path

(3) M has 1 more edge

case (2) must happen because $|M^*| > |M|$ □