COMPSCI 330: Design and Analysis of Algorithms **November 14, 2017**

Lecture 20: Duality

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20.1 Overview

In this lecture, we will introduce the notion of duality in linear programming. To do so, we will demonstrate how linear programming can be applied to solve for optimal strategies in two-player games. We will see that by considering the game from each player's perspective, we end up with two LPs describing the same problem. This result is the basic idea behind duality, and will be extended upon in future lectures.

20.2 Two-Player Zero-Sum Games

A two-player zero-sum game is a game played with two competing players, such that when one player wins, the other player loses. In studying these games, we often want to determine the optimal strategy for each player. A two-player zero-sum game can be represented by a matrix A, as follows. A contains a row for each action possible for player 1, and a column for each action possible for player 2. The entry $A[i,j]$ is then the payoff for player 1 if they select action i and player 2 selections action j. Because the game is zero-sum, $-A[i,j]$ is necessarily the payoff for player 2.

As an example, consider the classic game of Rock-Paper-Scissors. If we let the payoffs be 1 for a win, 0 for a draw, and -1 for a loss, then the matrix representation of the game is:

	R	P	S
R	0	-1	1
P	1	0	-1
S	-1	1	0

Players can either use a pure strategy or a mixed strategy. In a pure strategy, the player always selects the same action (i.e. they choose a single row/column of the matrix). This will clearly not be an effective strategy in Rock-Paper-Scissors, as it is easily countered (e.g. if player 1 always chooses rock, then player 2 can win by always choosing paper) In a mixed strategy, the player chooses each action according to some probability distribution (i.e. choose action 1 with probability p_1, action 2 with probability p_2, etc.). In the Rock-Paper-Scissors game, there is an optimal mixed strategy given by selecting each possible action with probability 1/3.

Finally, to calculate the payoff for a game, suppose that S_{row} is the optimal strategy for the row player, and S_{col} is the optimal strategy for the column player. The payoff for the row player (and so the negative of the column player's payoff) is:

$$\mathbb{E}_{i \sim S_{row}, j \sim S_{col}} A[i,j]$$

20.3 Solving Two-Player Games with LP

We will now show how linear programming can be used to solve for optimal strategies in two-player zero-sum games. Consider a game between two players, Duke and UNC, where each player can choose between 3 actions labeled A, B, and C. The game matrix is as follows, with Duke as the row player and UNC as the column player.

	A	B	C
A	3	1	-1
B	-2	3	2
C	1	-2	4

How can we use LP to find a good strategy for Duke? Let Duke play A with probability x_1, B with probability x_2, and C with probability x_3. We want to assign these probabilities in a way so that no matter what UNC chooses, Duke can be sure to get a good payoff of at least x_4. This gives the following LP:

$$\begin{aligned} \max x_4 \\ x_1, x_2, x_3 \geq 0 \\ x_1 + x_2 + x_3 = 1 \\ 3x_1 - 2x_2 + x_3 \geq x_4 \\ x_1 + 3x_2 - 2x_3 \geq x_4 \\ -x_1 + 2x_2 + 4x_3 \geq x_4 \end{aligned}$$

To understand this LP, note that we maximize x_4, as this represents the payoff Duke is ensured of, regardless of the action taken by UNC. The next two lines come from the fact that x_1, x_2, x_3 are probabilities. The last 3 lines reflect that the payoff for Duke must be at least x_4, regardless of the action UNC takes. The solution of this LP is:

$$x_1 = 9/19, x_2 = 6/19, x_3 = 4/19, x_4 = 1$$

Thus, by choosing A with probability 9/19, B with probability 6/19, and C with probability 4/19, Duke can guarantee a payoff of 1, no matter what UNC chooses.

Now let's approach this game from UNC's perspective. As before, let UNC play A with probability y_1, B with probability y_2, and C with probability y_3. We now want to assign these probabilities in a way so that no matter what Duke chooses, UNC can be sure that Duke's payoff is always low, at most y_4. This gives the following LP:

$$\begin{aligned} \min y_4 \\ y_1, y_2, y_3 \geq 0 \\ y_1 + y_2 + y_3 = 1 \\ 3y_1 + y_2 - y_3 \leq y_4 \\ -2y_1 + 3y_2 + 2y_3 \leq y_4 \\ y_1 + -2y_2 + 4y_3 \leq y_4 \end{aligned}$$

The solution for this LP is:

$$y_1 = 1/3, y_2 = 1/3, y_3 = 1/3, y_4 = 1$$

Thus, by choosing A with probability 1/3, B with probability 1/3, and C with probability 1/3, UNC can guarantee that Duke's payoff is no higher than 1, no matter what action they choose.

In general, we see that Duke is guaranteed to get at least x_4 points, and UNC can guarantee that Duke gets no more than y_4 points. It is clear then that we must have $x_4 \leq y_4$. This is known as the weak duality condition. For two-player zero-sum games, however, Von Neumann found that there is always a pair of optimal strategies and a single value V, such that if the row player plays optimally, they are guaranteed a payoff of at least V, and if the column player plays optimally, then they guarantee a payoff of at most V. This indicates that the solutions to our two LPs are equal, i.e. $x_4 = y_4$. This is known as strong duality.

Future lectures will build upon these results, and extend them to general linear programming problems.