Fun with Reductions

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Dec 5, 2017

Outline

Show

- Vertex 3-Coloring
- Hamiltonian Cycle
- Super Mario

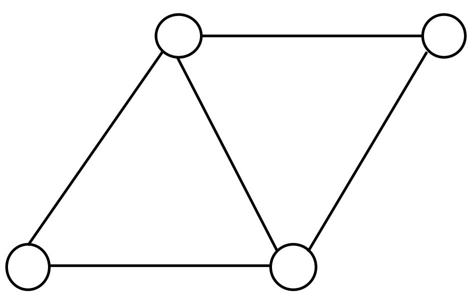
are NP-Hard.

Recap

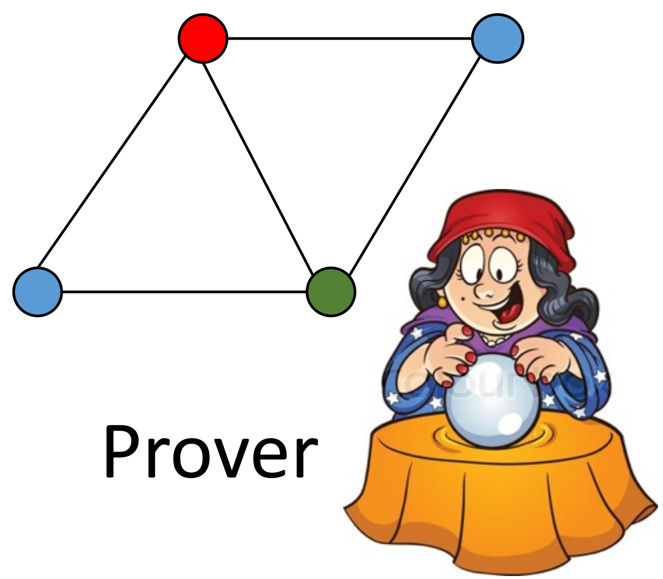
- P vs NP
- (Polynomial-time) Reductions
- 3-Satisfiability (3SAT)

"Easy to verify" problems: NP

- All decision problems such that we can verify the correctness of a solution in polynomial time.



input

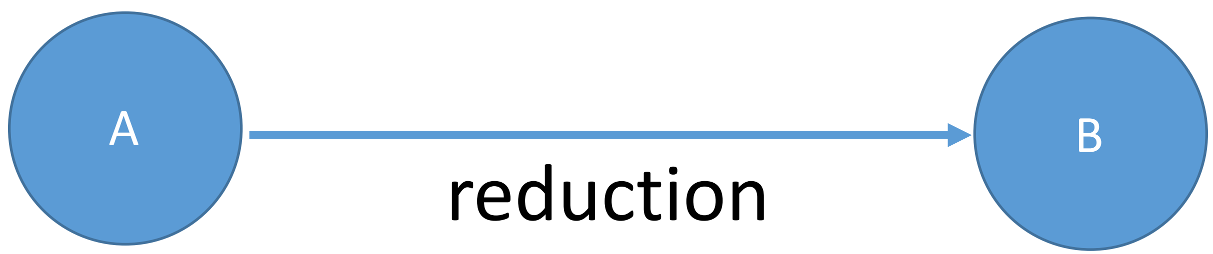


Prover

Verifier: OK, that is indeed a solution.

Polynomial time reductions

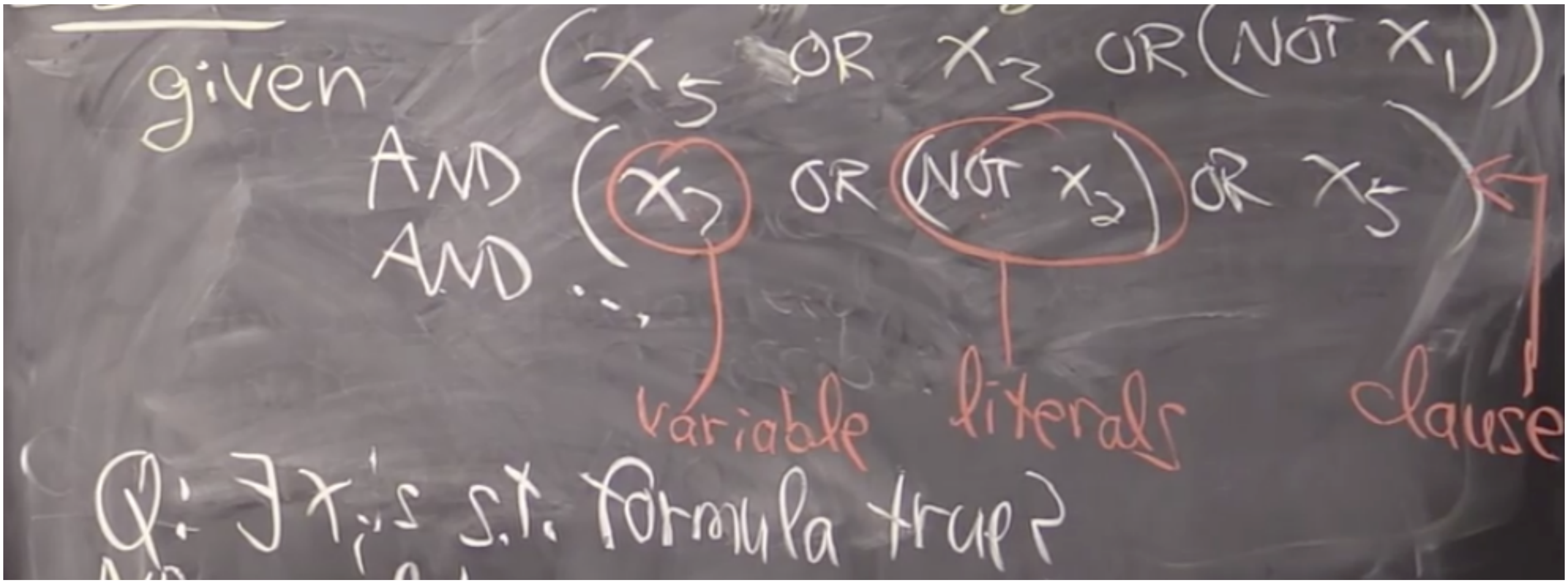
- Reduce A to B: a polynomial time algorithm that maps instances of A to instances of problem B, such that the answers are the same.



- $A \leq_p B$: B is at least as hard as A.

  If you can solve B (in poly time) then you can solve A.

3-Satisfiability (3SAT)



Gadget-Based Reductions

$A \leq_p B$:

Given instances of A, output instances of B.

Build gadgets for pieces of A.

Put the pieces together.

$$3\text{SAT} \leq_p X$$

Fun with Hardness Proofs

Algorithmic Lower Bounds:
Fun with Hardness Proofs

Erik Demaine

http://courses.csail.mit.edu/6.890/fall14/lectures/

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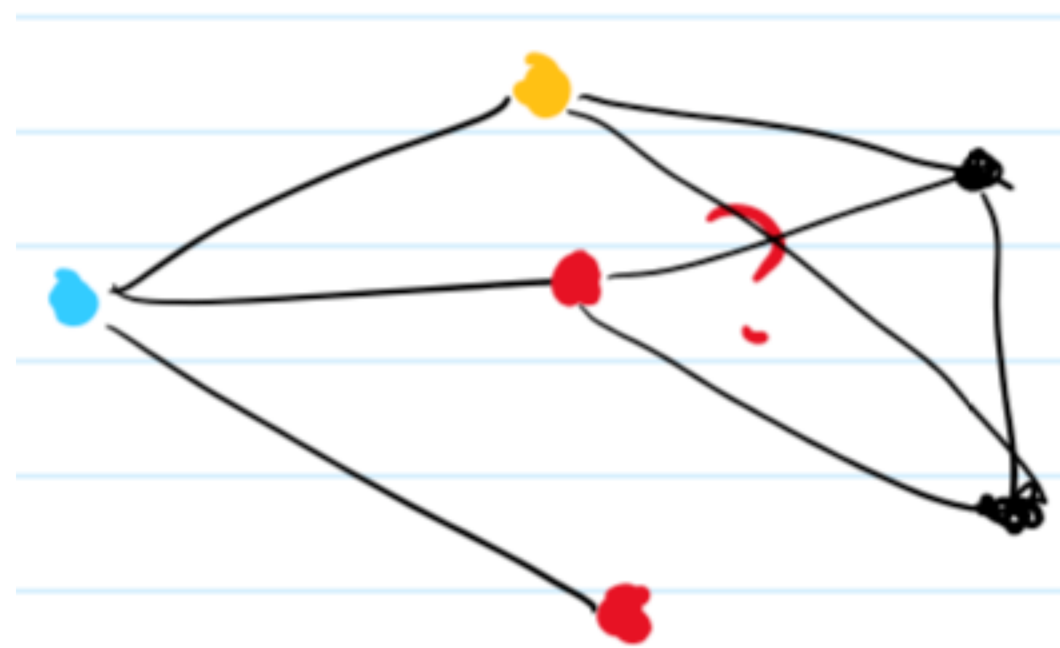
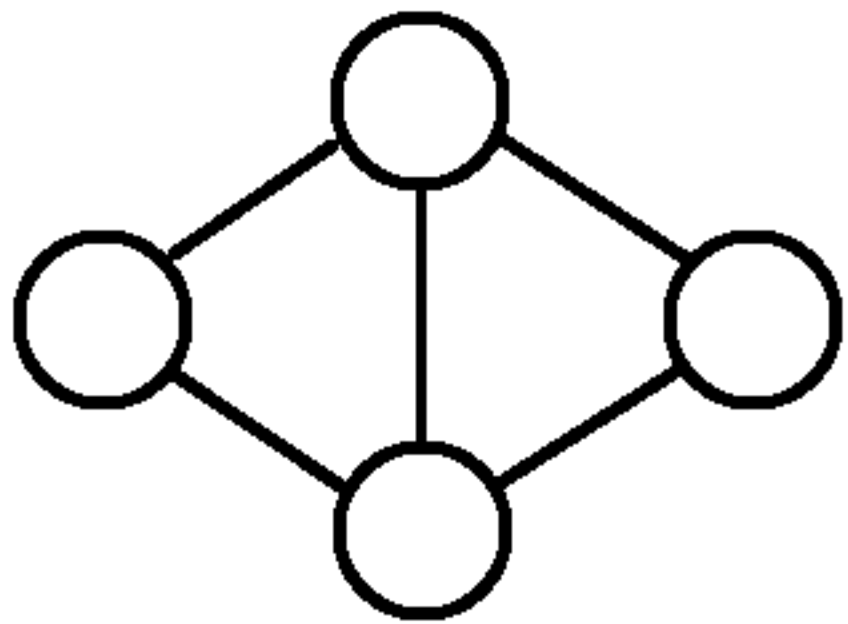
Vertex 3-Coloring



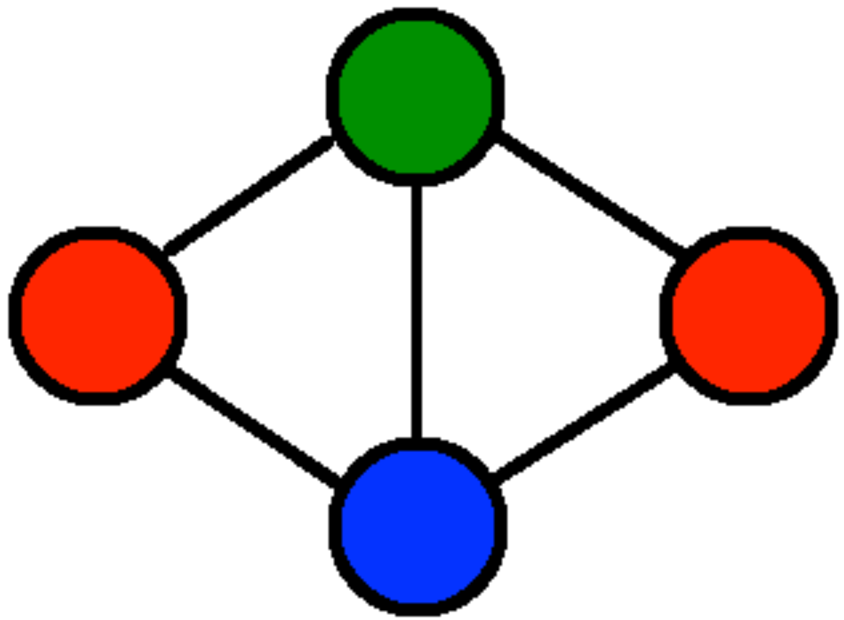
Input: a graph

Output: color each vertex using 1 of the 3 colors, so that adjacent vertices do not get the same color.

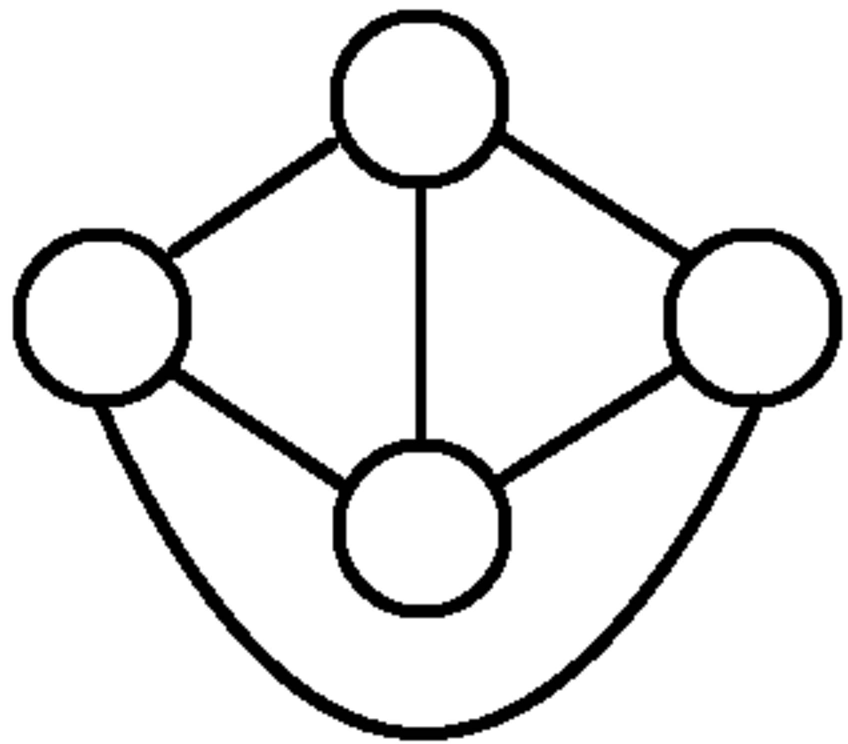
3-Coloring:



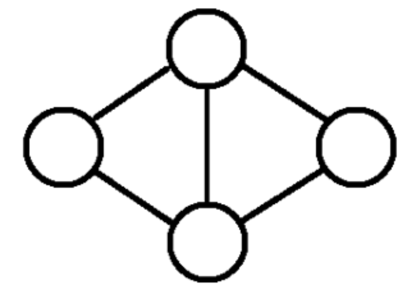
3-Coloring: Yes instance

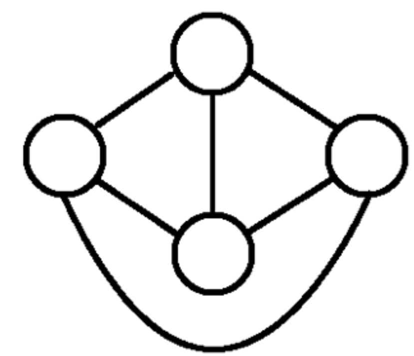


3-Coloring: No instance



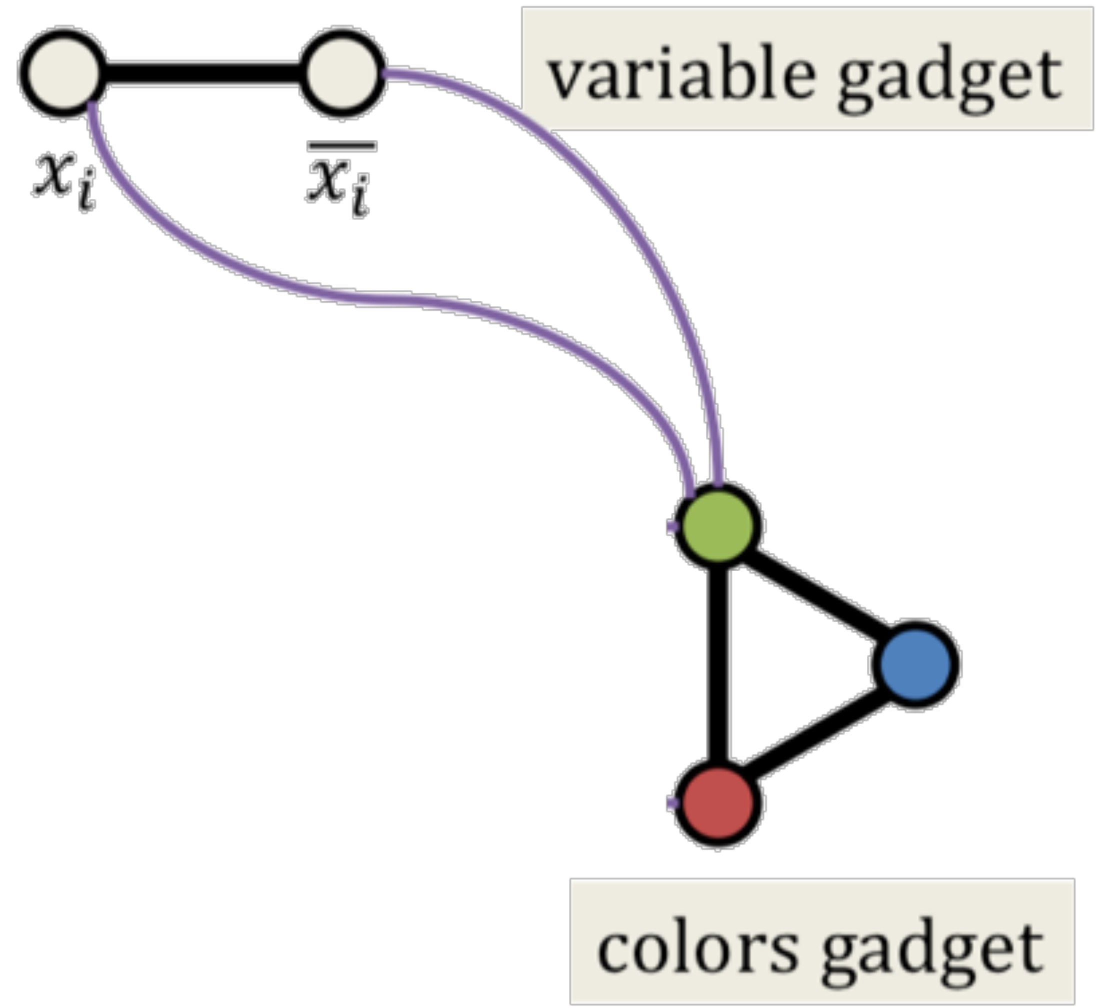
3SAT $\leq_p$ 3-Coloring

Satisfiable formula $\Longleftrightarrow$ 

Unsatisfiable formula $\Longleftrightarrow$ 

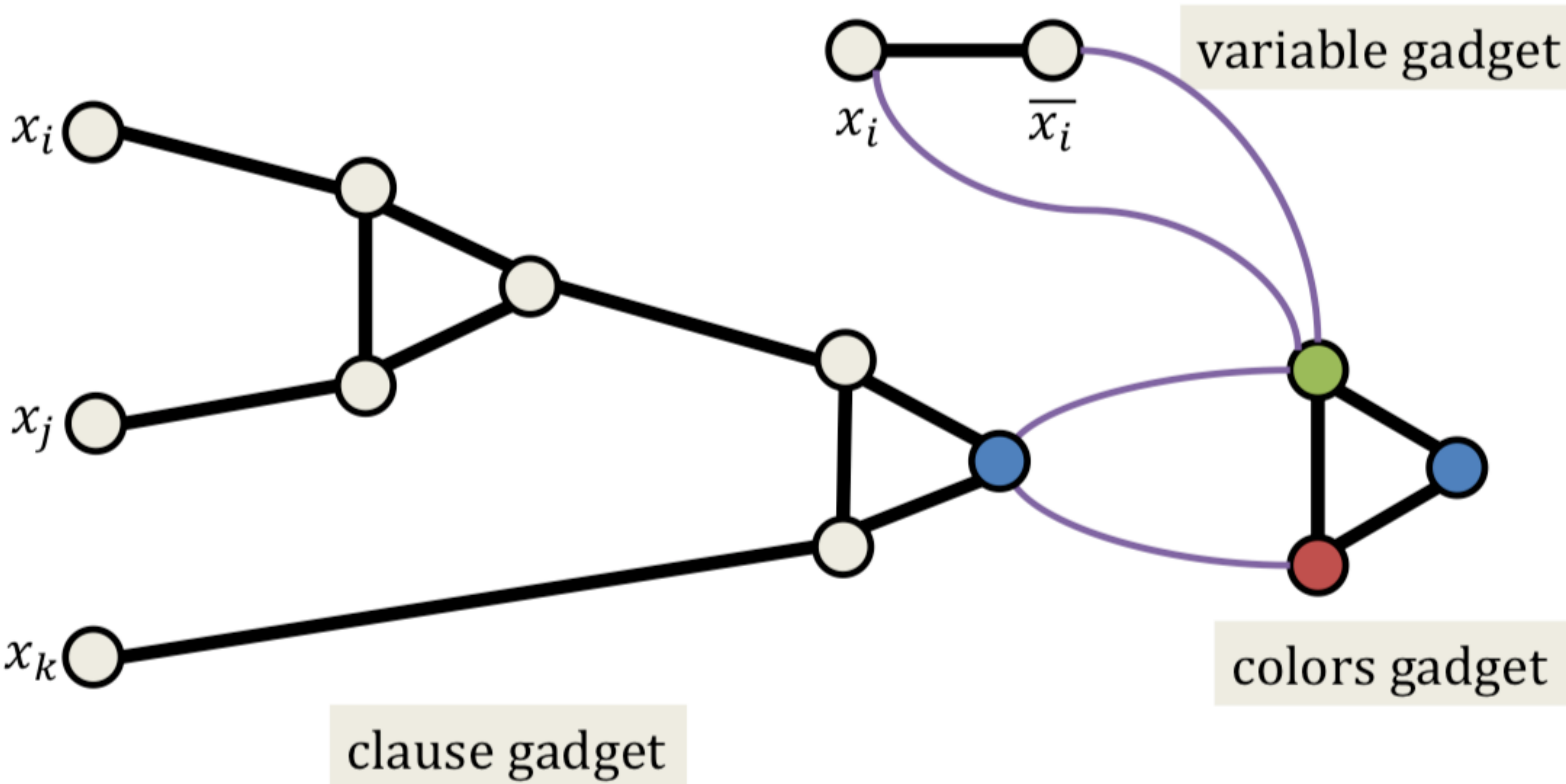
Vertex 3-Coloring

[Garey, Johnson, Stockmeyer 1976]



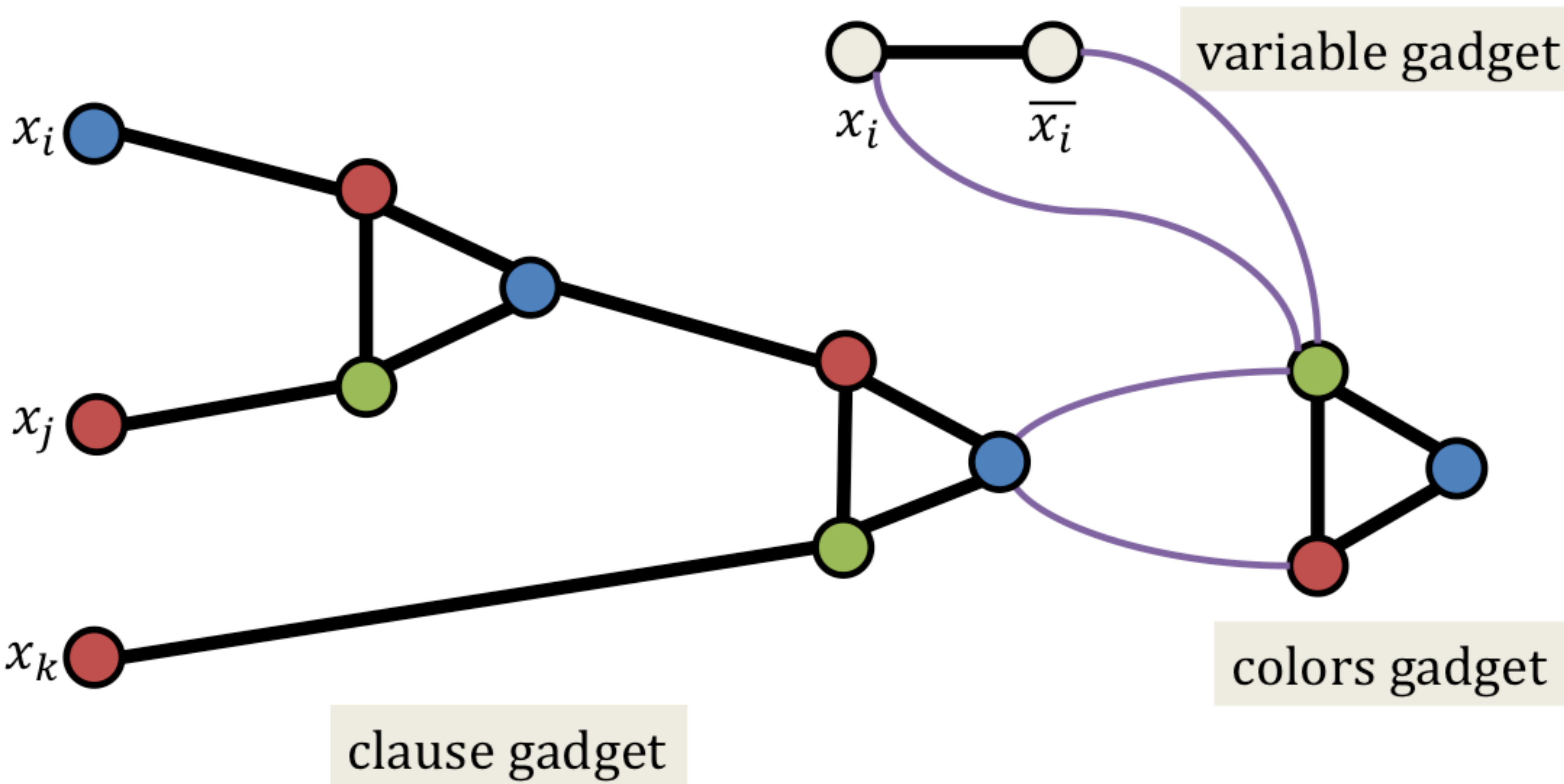
Vertex 3-Coloring

[Garey, Johnson, Stockmeyer 1976]



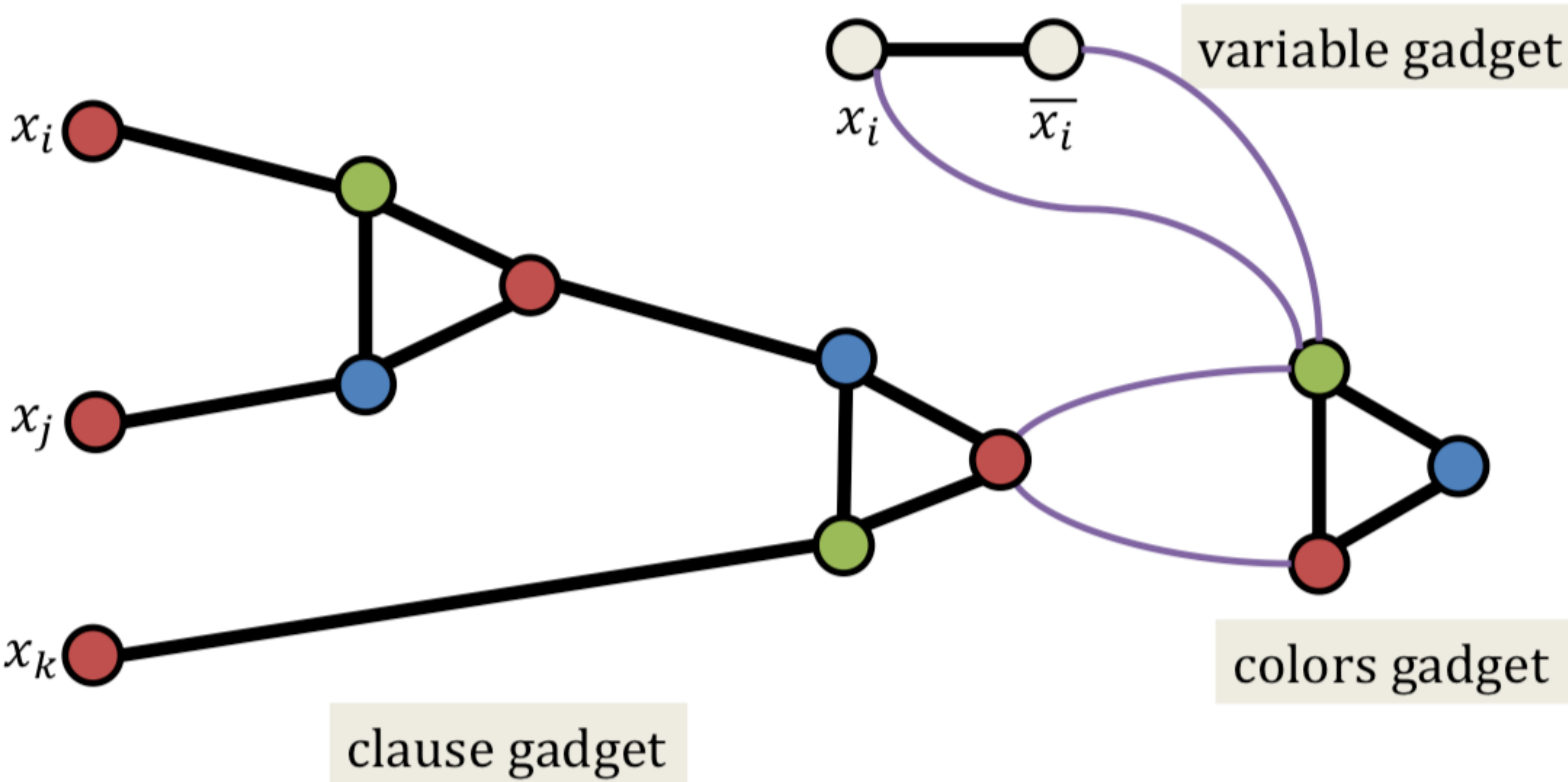
Vertex 3-Coloring

[Garey, Johnson, Stockmeyer 1976]



Vertex 3-Coloring

[Garey, Johnson, Stockmeyer 1976]



3SAT $\leq_p$ 3-Coloring

- Consequence:
  3-Coloring is NP-Complete.
  (Because 3-Coloring is also in NP.)

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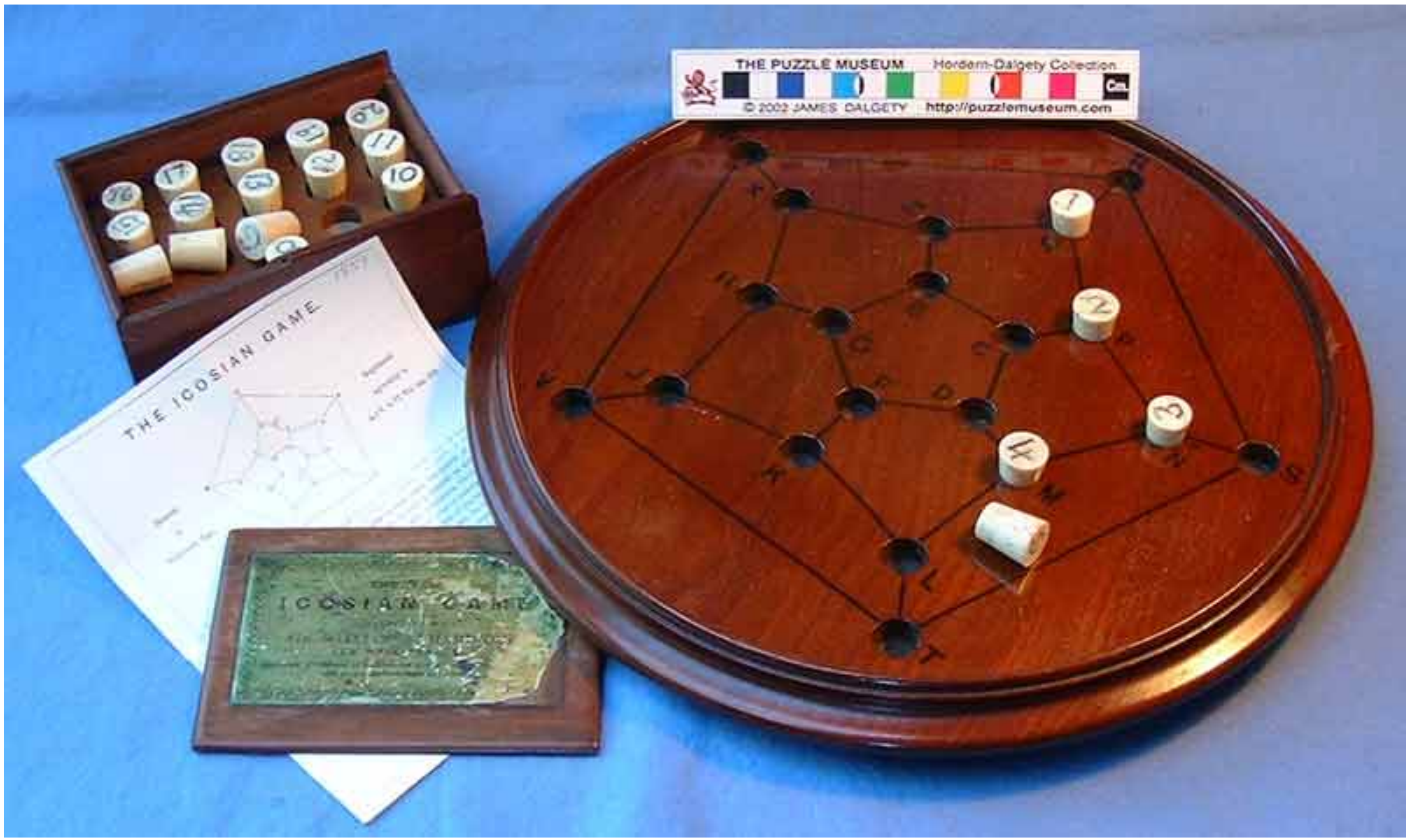
Hamiltonian Cycle

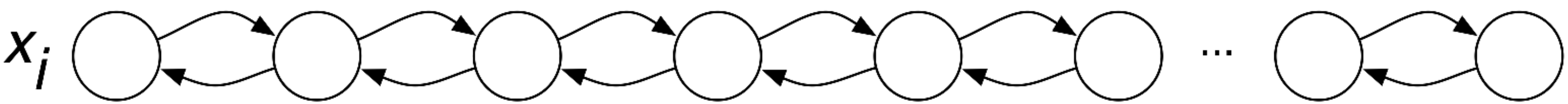


Hamiltonian Cycle

- Input: a (directed) graph.

- Solution: a cycle visiting every vertex exactly once.

Variable Gadget



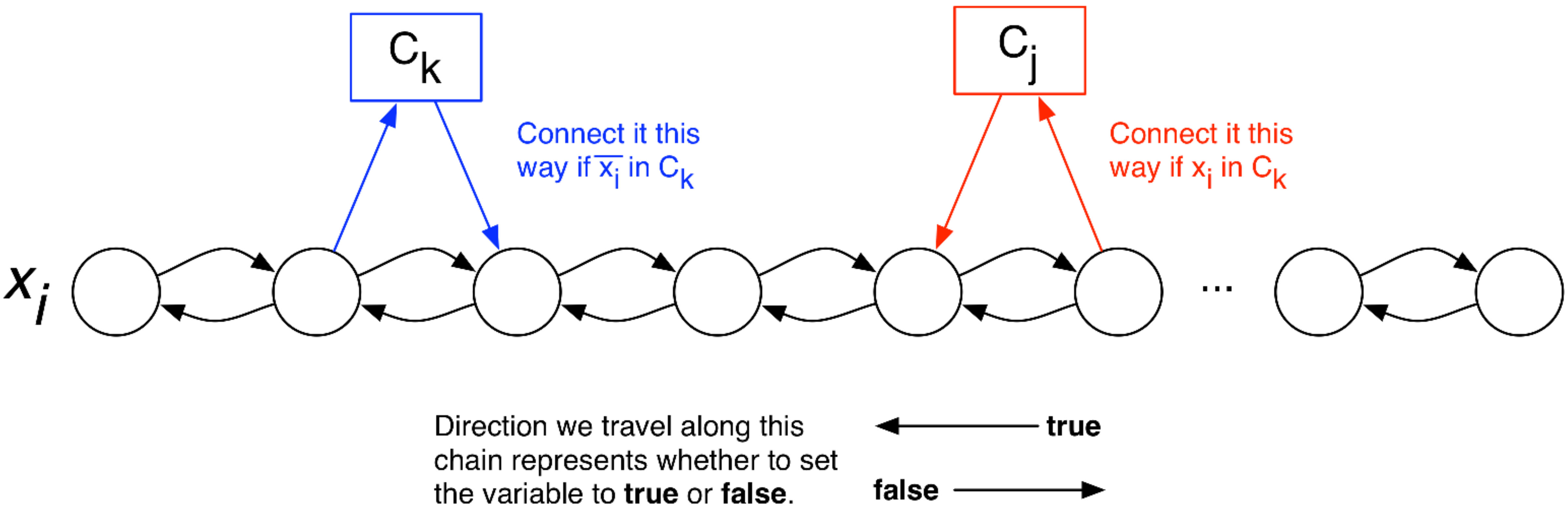
Direction we travel along this chain represents whether to set the variable to **true** or **false**.

← **true**

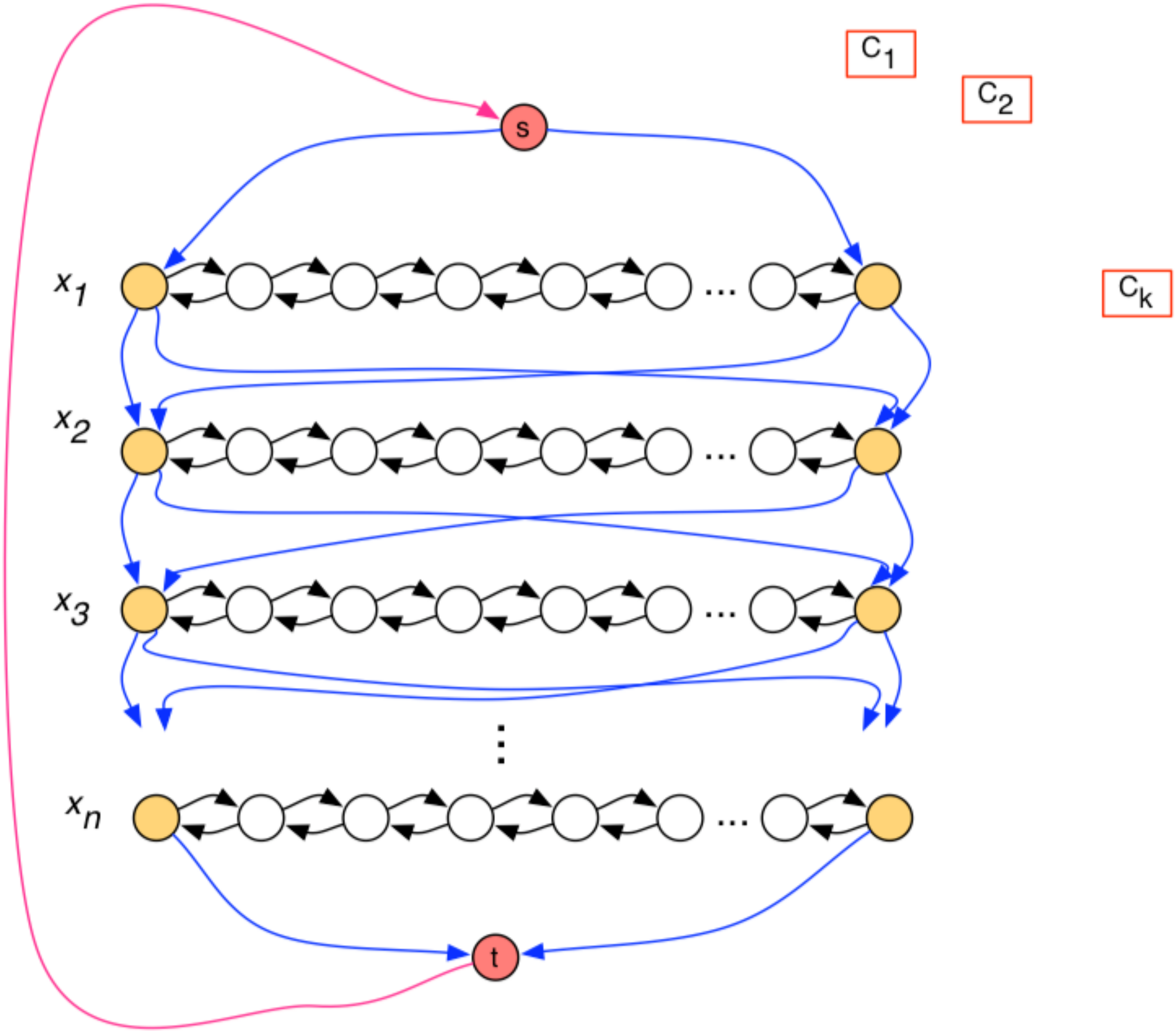
false →

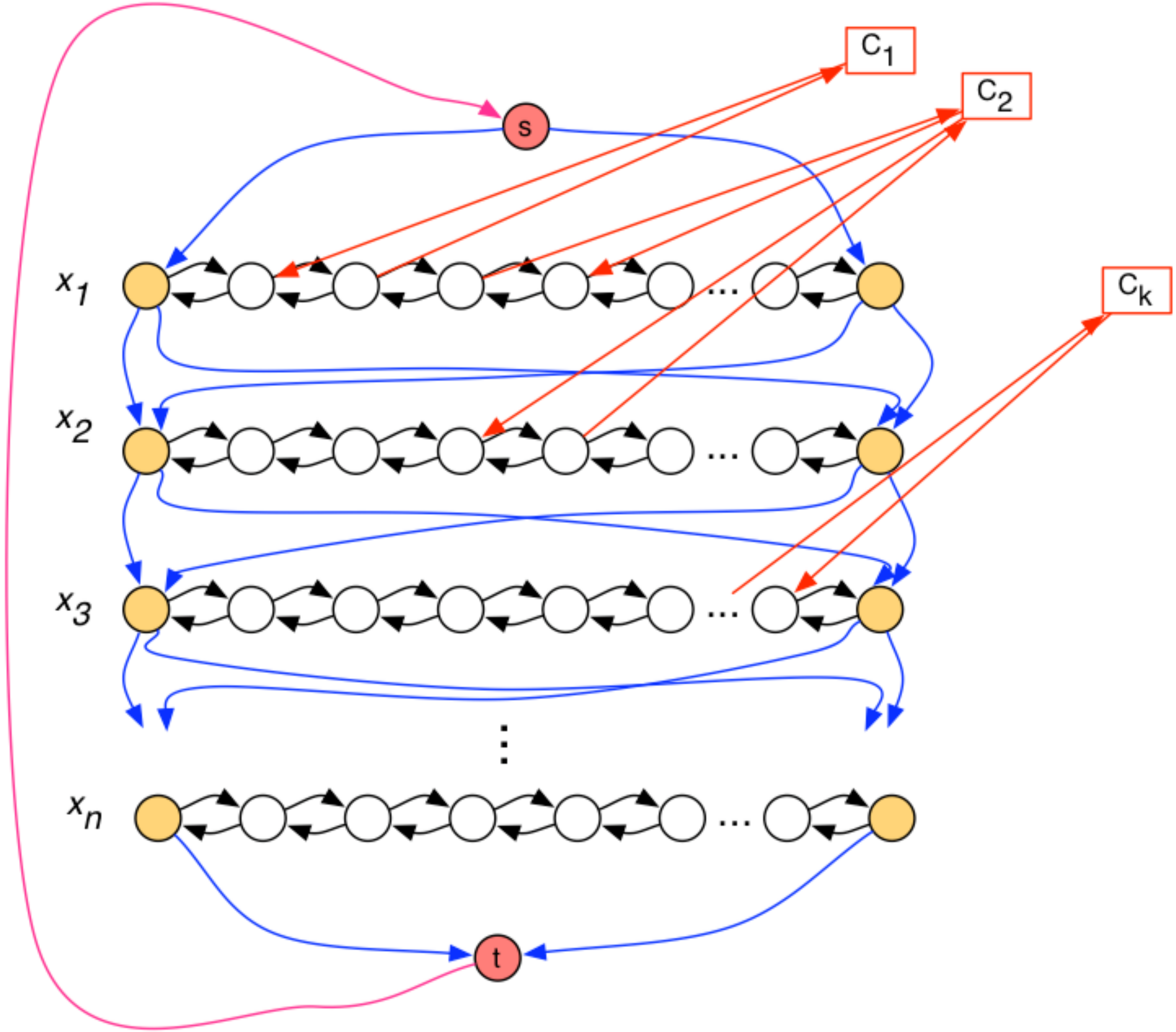
Clause Gadget

Add a new node for each clause:



3SAT $\leq_p$ Hamiltonian Cycle

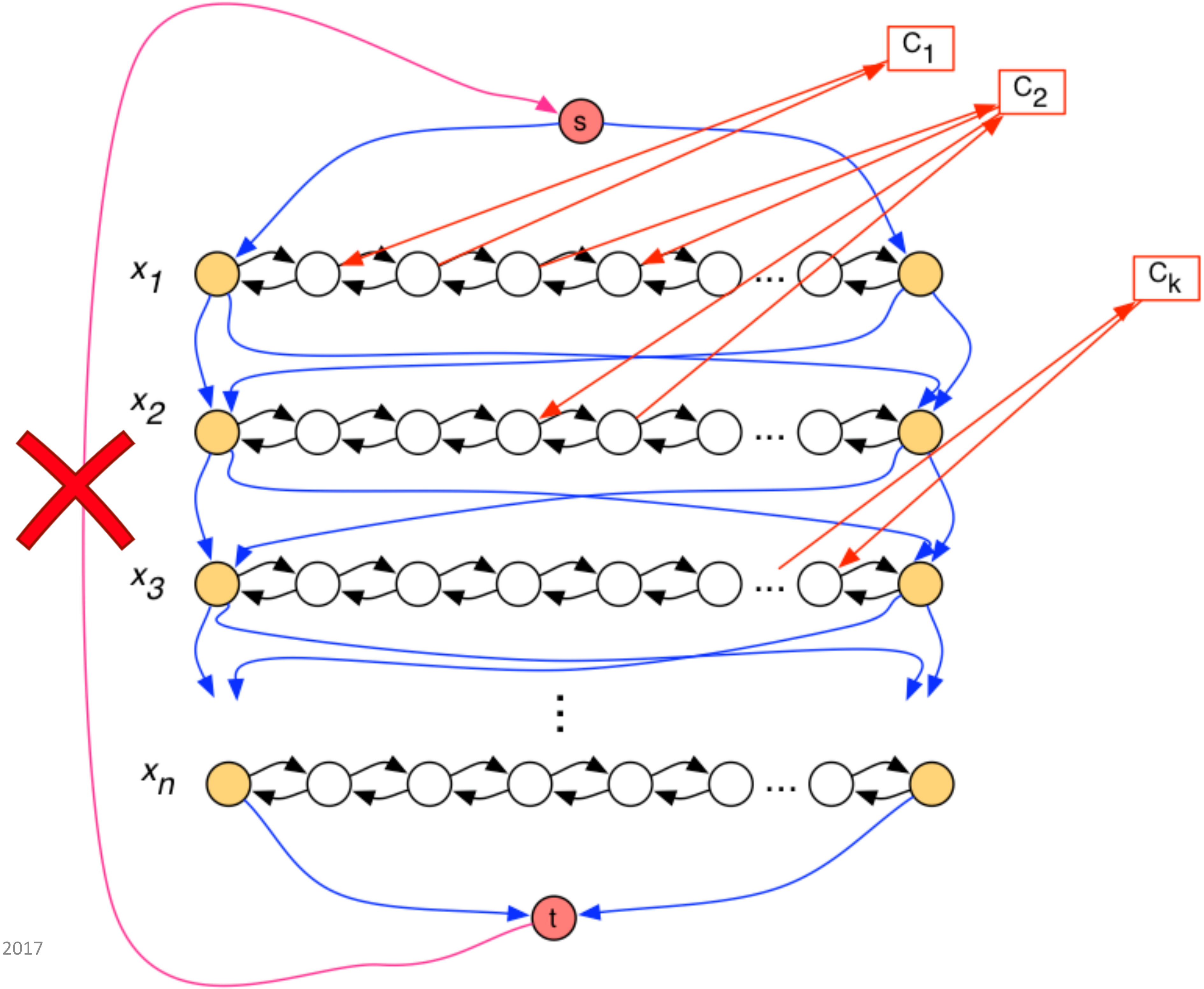

C1
C2
Ck
s
x1
x2
x3
xn
t



C1
C2
Ck
s
x1
x2
x3
xn
t

$3SAT \leq_p$ Hamiltonian Cycle ✓

$3SAT \leq_p$ Hamiltonian Path ?



C1
C2
Ck
s
x1
x2
x3
xn
t

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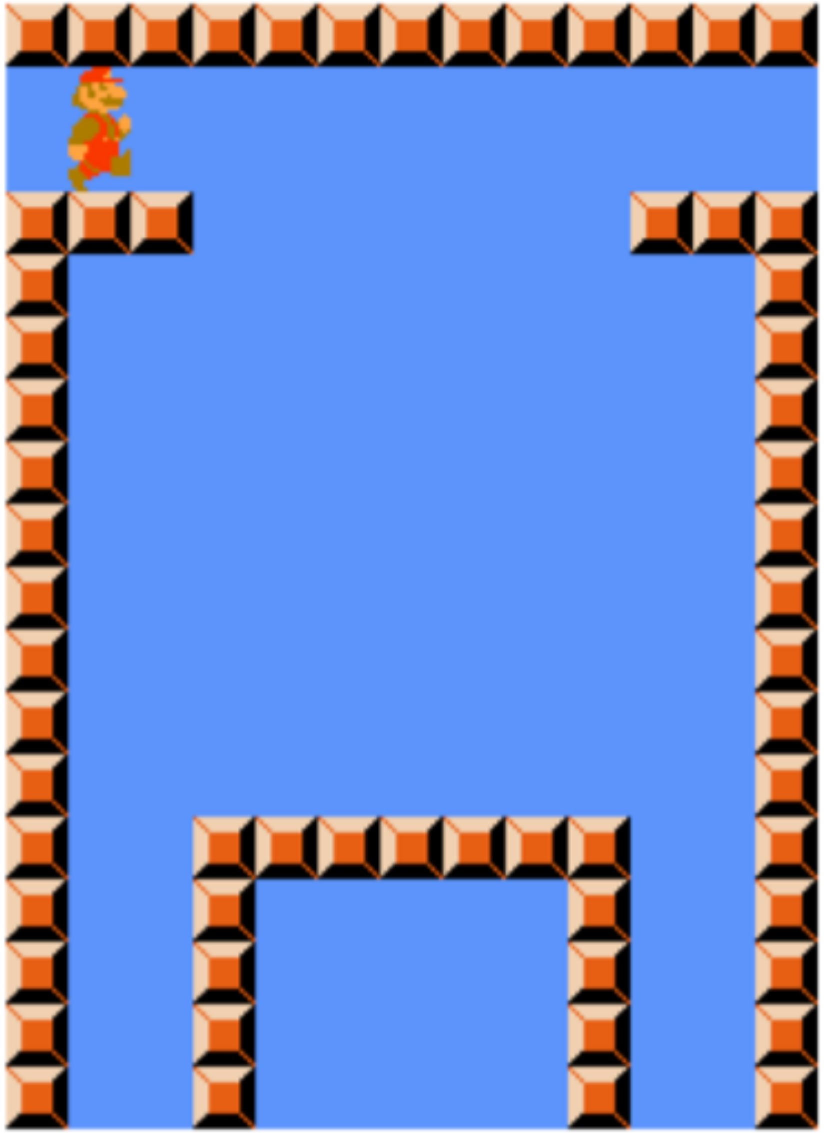
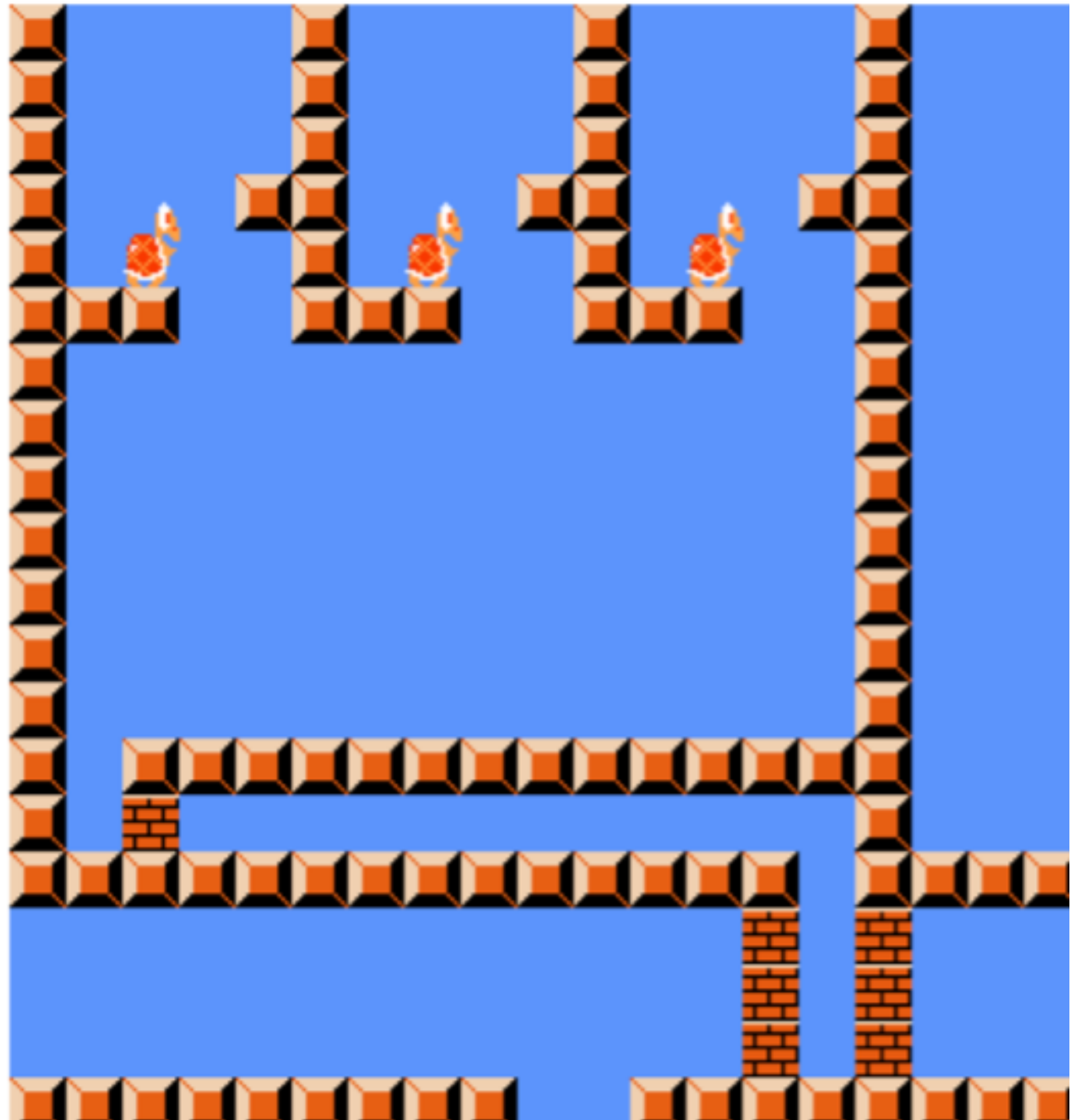
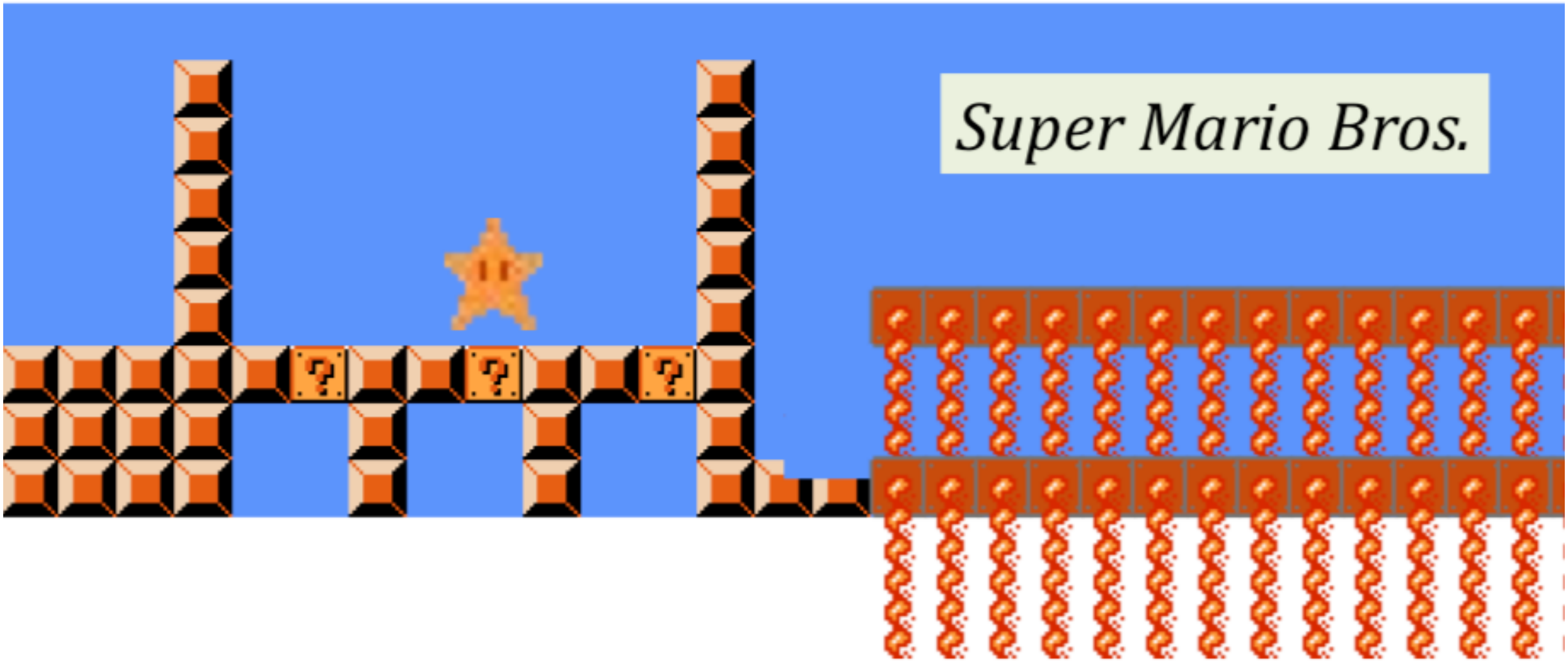
Super Mario Bros. is NP-Hard

[Aloupis, Demaine, Guo 2012]



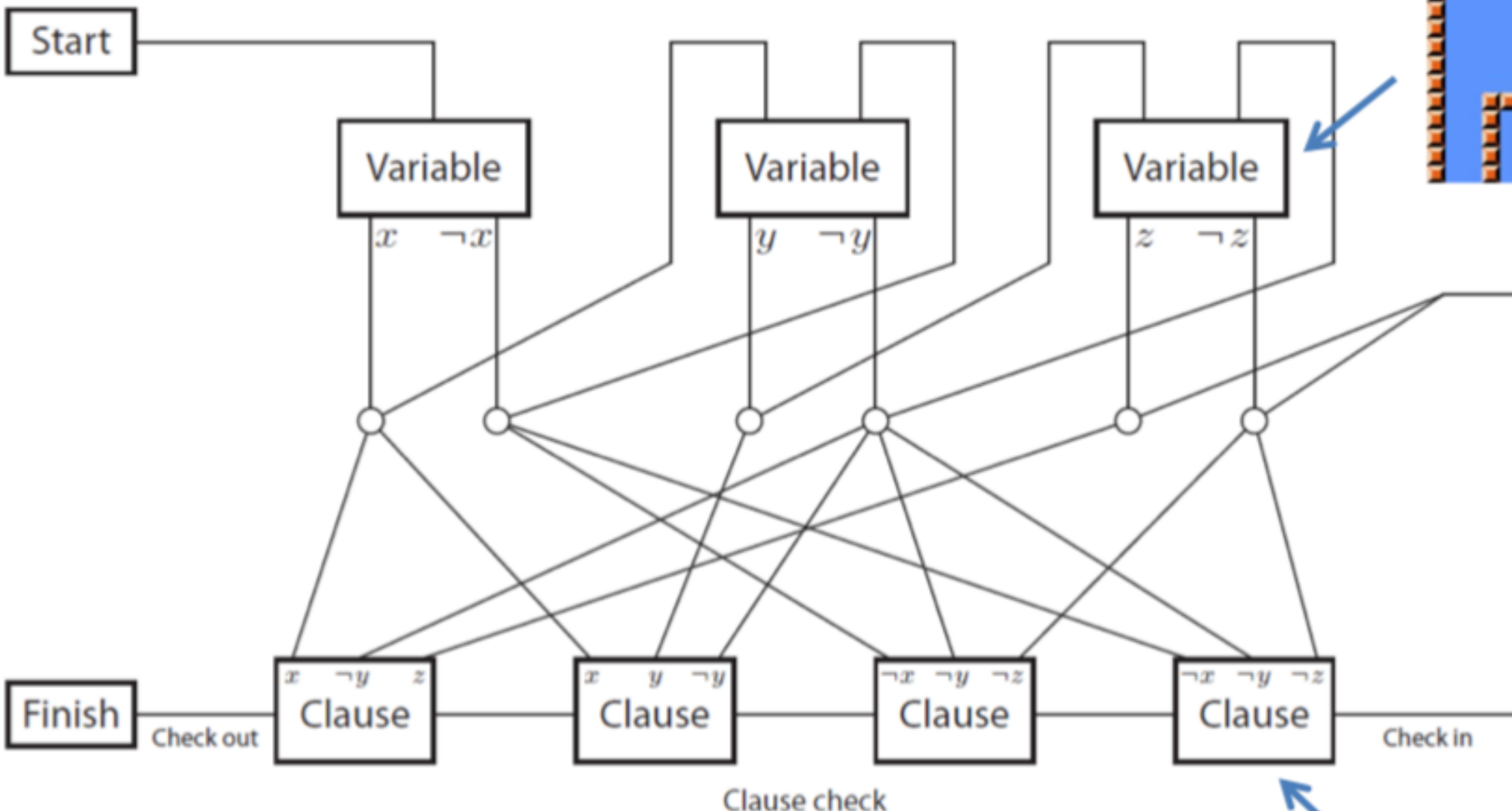


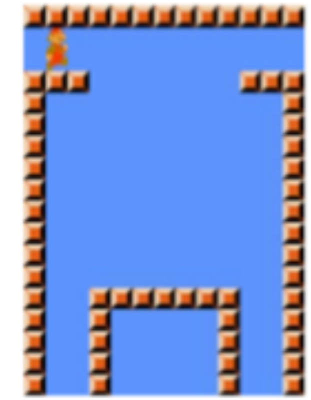
clause



Super Mario Bros. is NP-Hard

[Aloupis, Demaine, Guo, Viglietta 2014]

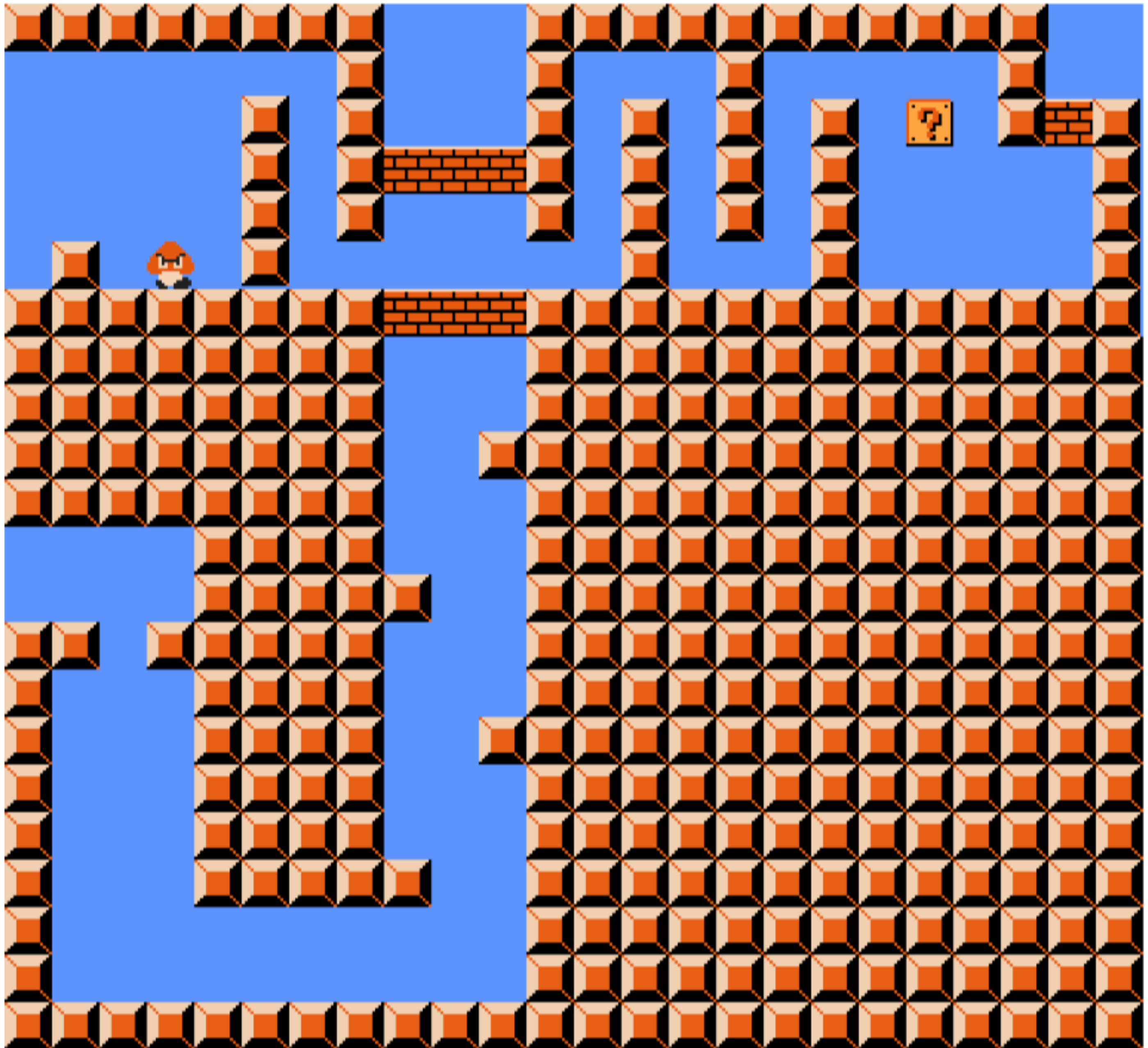




$(x \text{ OR } \neg y \text{ OR } z)\ \&\ (x \text{ OR } y \text{ OR } \neg y)\ \&$
$(\neg x \text{ OR } \neg y \text{ OR } \neg z)\ \&\ (\neg x \text{ OR } \neg y \text{ OR } \neg z)$

Super Mario Bros. is NP-Hard

[Aloupis, Demaine, Guo, Viglietta 2014]



crossover