COMPSCI 632: Approximation Algorithms	Oct 11, 2017
Lecture 12	
Lecturer: Debmalya Panigrahi	*Scribe: Hanrui Zhang*

1 Overview

This lecture covers iterative rounding techniques as a tool to show intergrality of extreme point LP solutions.

2 Structure of Extreme Point LP Solutions

Before we start, observe the following straightforward structural property of extreme point solutions:

Lemma 1. *In an extreme point solution, there are exactly n linearly independent constraints that are simultaneously tight, where n is the number of variables.*

Also, note that the property holds regardless of the objective.

3 Maximum Weight Matching

An LP formulation. Consider the following LP:

$$\begin{aligned} \text{maximize:} \quad & \sum_{e \in E} w_e x_e \\ \text{subject to:} \quad & \sum_{e \in \delta_v} x_e \leq 1, \forall v \in V \\ & x_e \geq 0, \forall e \in E. \end{aligned}$$

Theorem 2. *All extreme point solutions of the LP formulation above are integral.*

Proof. To show integrality, we consider the following "elimination" procedure.

1. Take any extreme point solution.
2. While there is some e such that $x_e = 0$, discard e. The remaining variables form an extreme point solution to the LP induced by the remaining graph.
3. Now $x_e > 0$ for any e. We claim that in the remaining graph, $|E| \leq |V|$: There are $|E|$ variables and at least $|E|$ tight constraints. All tight constraints correspond to vertices. The number of tight constraints cannot exceed the total number of constraints which can be tight, namely $|V|$.
4. While there is some e such that $x_e = 1$, add e to the solution and remove e and its end points. What remains is still an extreme point solution, and the old solution is integral iff the new solution is integral.
5. Now $|E| = 0$, and the solution is trivially integral.

We still need to show that at step 4, whenever $|E| > 0$, there is some e such that $x_e = 1$. Let V' be the set of tight vertices. Consider two cases:

- $|E| \leq |V'| - 1$. There must be some tight vertex with degree 1, say v^*. (Suppose every $v \in V'$ has degree at least 2. We have $2|V'| - 2 \geq 2|E| = \sum_{v \in V'} deg(v) \geq 2|V'|$, a contradiction.) The only edge incident on v^*, e^*, must satisfy $x_{e^*} = 1$.

- $|E| = |V'|$. Either there is some vertex with degree 1, or all vertices in V' have degree 2. The former reduces to the first bullet point. When the latter happens, we have a non-empty disjoint family of cycles. Bipartiteness requires each of these cycles to be even. Take any of these cycles, it is easy to see that the constraints induced by vertices in the cycle are not independent, which contradicts Lemma 1.

□

Again, our entire argument has nothing to do with the objective.

Generalization to non-bipartite graphs. The foregoing LP formulation is not necessarily integral for non-bipartite graphs. To handle general graphs, consider the following LP:

$$
\begin{aligned}
\text{maximize:} \quad & \sum_{e \in E} w_e x_e \\
\text{subject to:} \quad & \sum_{e \in S \times S} x_e \leq \left\lfloor \frac{|S|}{2} \right\rfloor, \forall S \subseteq V \\
& x_e \geq 0, \forall e \in E.
\end{aligned}
$$

It turns out that this formulation is integral and separable.

4 Minimum Spanning Tree

A naive LP formulation. Consider the following LP:

$$
\begin{aligned}
\text{minimize:} \quad & \sum_{e} w_e x_e \\
\text{subject to:} \quad & \sum_{e \in \delta(S)} x_e \geq 1, \forall S \subseteq V \\
& x_e \geq 0, \forall e \in E.
\end{aligned}
$$

This LP unfortunately has an integrality gap of 2: Consider a long ring with equal edge weights. An MST consists of any $n-1$ edges, whereas an optimal solution to the LP picks half of each edge.

To fix this LP, one may modify the constraints for connectivity to be:

$$
\sum_{e \in \delta(\pi)} x_e \geq |\pi| - 1, \forall \text{partition } \pi.
$$

This new LP is valid and integral. We will not discuss further on this formulation.

The subtour formulation for MST. Consider the following LP:

$$\begin{aligned}
\text{minimize:} \quad & \sum_e w_e x_e \\
\text{subject to:} \quad & \sum_{e \in S \times S} x_e \leq |S| - 1, \forall S \subseteq V \\
& \sum_{e \in E} x_e = n - 1 \\
& x_e \geq 0, \forall e \in E.
\end{aligned}$$

We will show that this LP is integral. First note that:

Lemma 3. *If for $S, T \subseteq V$, we have*

$$\sum_{e \in S \times S} x_e = |S| - 1,$$

$$\sum_{e \in T \times T} x_e = |T| - 1.$$

Then it is also true that

$$\sum_{e \in (S \cap T) \times (S \cap T)} x_e = |S \cap T| - 1,$$

$$\sum_{e \in (S \cup T) \times (S \cup T)} x_e = |S \cup T| - 1.$$

Proof. Let

$$x(V') = \sum_{e \in V' \times V'} x_e.$$

It is easy to see that

$$x(S \cup T) + x(S \cap T) = x(S) + x(T) + \sum_{e \in S \setminus T \times T \setminus S} x_e \geq x(S) + x(T) = |S| + |T| - 2.$$

On the other hand,

$$|S| + |T| - 2 = |S \cup T| + |S \cap T| - 2 \geq x(S \cup T) + x(S \cap T) \geq |S| + |T| - 2.$$

All inequalities must be tight. It follows immediately that

$$x(S \cup T) = |S \cup T| - 1,$$

$$x(S \cap T) = |S \cap T| - 1.$$

□

Consequently we have this lemma:

Lemma 4. *Let $\mathscr{L}$ be a maximal laminar family in the family of tight sets $\mathscr{S}$, then*

$$\text{span}(\mathscr{L}) = \text{span}(\mathscr{S}).$$

The rest of the proof will be given in the next lecture.