COMPSCI 632: Approximation Algorithms	October 16, 2017
Lecture 13	
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1 Overview

Last time we looked at iterative rounding techniques for the Maximum weight matching problem and the subtour formulation of the MST problem. In this lecture, we introduce a local token counting argument and apply it to degree-bounded MST.

2 MST (subtour LP) and Local Token Counting Argument

From last lecture, we know that the naive LP for MST has an integrality gap of 2. Instead, consider the following subtour LP formulation for MST:

$$\begin{aligned}
\text{minimize:} \quad & \sum_e w_e x_e & \\
\text{subject to:} \quad & \sum_{e \in S \times S} x_e \leq |S| - 1 & , \forall S \subset V \\
& \sum_{e \in E} x_e = |V| - 1 & \\
& x_e \geq 0 & , \forall e \in E
\end{aligned}$$

Take an extreme point solution for the above LP. If there exists an edge $e \in E$ with $x_e = 0$ or $x_e = 1$, we are guaranteed to make progress in the "elimination" procedure to get to an integral solution. But could it be that $x_e \in (0,1)$ for every edge e? In this case, we rely on the property of extreme point solution, i.e., $|\mathscr{L}| = |E| = m$ where $\mathscr{L}$ is the maximal laminar family of tight sets, to get a contradiction.

Lemma 1. *Let $\mathscr{L}$ be the maximal laminar family of tight sets and n be the number of vertices in V. Then, $|\mathscr{L}| \leq n-1$.*

Proof. We prove this by induction on n. When $n = 2$, there can be at most one tight set, i.e., the set containing both the vertices. Thus, $|\mathscr{L}| \leq 1$.
Consider a general set A of size n that contains a set B of size n'. We know that,

$$\begin{aligned}
|\mathscr{L}(A)| &\leq 1 + |\mathscr{L}(B)| + |\mathscr{L}(A \setminus B)| \\
&\leq 1 + |B| - 1 + |A| - |B| - 1 \\
&= |A| - 1
\end{aligned}$$

$\square$

As a result, $|\mathscr{L}| = m \leq n-1$ but $x(v) = |V| - 1$ Therefore, all remaining edges have $x_e = 1$. The MST polytope is integral.

2.1 Local Token Counting Argument

Another way to go about the analysis is to design a token assignment scheme that can be used to get a contradiction to $|\mathscr{L}| = |E|$, under the assumption that none of the conditions needed to make progress are satisfied.

Assume that $x_e \in (0,1)$ for each edge $e \in E$. Suppose each edge gets one token. Edge (u,v) gives x_{uv} tokens to the smallest set in $\mathscr{L}$ that contains both the vertices, u and v.

Claim. Not all tokens are distributed.

Proof. We know that $x_e \in (0,1), \forall e \in E$ and an edge e holds on to $1 - x_e$ tokens. □

Claim. Every set in $\mathscr{L}$ gets at least one token.

Proof. Consider a set A in $\mathscr{L}$. Let R_1, R_2, ..., R_k be the largest sets contained in A. Let us call these child sets. The possibilties for edges are: 1) edge has both endpoints in some child of A, 2) edge has one endpoint in a chid set and another in A, 3) edge has one endpoint in A and another outside of A. We define $x(A) := \sum_{e \in A \times A} x_e$, so

$$\begin{aligned} x(A) - \sum_{i=1}^{k} x(R_i) &= |A| - 1 - \sum_{i=1}^{k} (|R_i| - 1) \\ &= |A| - \sum_i |R_i| + k - 1 \\ &\geq 0 \end{aligned}$$

The above quantity cannot be negative because $x(A) \geq \sum\limits_{i=1}^{k} x(R_i)$ by definition. Also, it cannot be zero because the incidence vectors for the sets in $\mathscr{L}$ must be linearly independent. Note that $|A| - \sum_i |R_i| + k - 1$ must be an integer. Hence, each set in the maximal laminar family gets a non-zero, integral number of tokens. □

3 Degree-bounded MST

Let W be the set of vertices with degree-constraints. For a vertex $v \in W$, the degree-bound us given by B_v.

$$\begin{aligned} \text{minimize:} \quad & \sum_e w_e x_e \\ \text{subject to:} \quad & x(S) \leq |S| - 1, \forall S \subset V \\ & x(V) = |V| - 1 \\ & x(\delta(v)) \leq B_v, \forall v \in W \\ & x_e \geq 0, \forall e \in E \end{aligned}$$

In the case that B_v is 2 for all vertices, it represents the Hamiltonian path problem which is known to be NP-hard. The goal for this problem is to design an approximation algorithm that violates/relaxes the degree bounds but matches the cost to that of an optimal solution.

3.1 $(1, B_v + 2)$-approximation

Let F be the set of edges that get picked. F is empty initially. The algorithm does the following:

1. Solve LP.
2. For every edge e with $x_e = 0$, remove e.
3. For every leaf vertex v, add $e = (u, v)$ to F and update B_u.
4. Else, there exists a vertex v with $d_E(v) \leq 3$. Remove such vertices from W, i.e., drop the corresponding degree-bounds.

Next, we see why one of these conditions must be true. Note that we now have two types of tight constraints: 1) tight degree-constraints, and 2) usual laminar family of tight constraints (that come from the connectivity/MST requirements). Thus, $|\mathscr{L}| + |T| = |E| = m$, where T is the set of tight degree-constraints. Suppose none of the above conditions are met. This means that $d_E(v) \geq 4$ for every $v \in W$ and total degree at these vertices is at least $4|T|$. Then,

$$|E| \geq \frac{2|V \setminus T| + 4|T|}{2} = |V \setminus T| + 2|T| = |V| + |T|$$
$$\implies |E| - |T| = |\mathscr{L}| \geq |V| = n$$

But $|\mathscr{L}|$ cannot exceed $n-1$ (using Lemma 1). This gives us a contradiction!

3.2 $(1, B_v + 1)$-approximation

Let F be the set of edges that get picked. F is empty initially. The algorithm does the following:

1. Solve LP.
2. For every edge e with $x_e = 0$, remove e.
3. Else, there exists a vertex v with $d_E(v) \leq B_v + 1$. Remove such vertices from W, i.e., drop the corresponding degree-bounds.

We prove that the algorithms is able to make progress through a proof by contradiction. Suppose none of the required conditions are met. We give one token to each edge that get distributed according to the following token assignment scheme:

- The smallest set in $\mathscr{L}$ that contains both endpoint of an edge e gets x_e tokens.
- $\forall e = (u, v) \in E$, u and v both get $\frac{1-x_e}{2}$ amount of tokens.

Claim. Every set in $\mathscr{L}$ gets at least one token.
(We proved this in section 2.1).

Claim. Every tight vertex in T gets at least one token.

Proof. Let v be a vertex in T, i.e., $x(\delta(v)) = B_v$. Also, remember that $\delta_E(v) \geq B_v + 2$. Then for v,

$$\sum_{e \in \delta(v)} \frac{1-x_e}{2} = \frac{|\delta(v)|}{2} - \frac{x(\delta(v))}{2} = \frac{|\delta(v)|}{2} - \frac{B_v}{2} \geq 1.$$

□

This suggests that the total number of tokens is at least $|\mathscr{L}|+|T|=|E|$. But are all the tokens distributed?

1. $V \neq T$, then tokens are left behind: if there exists a $v \notin T$ then it gets tokens that have not been accounted for yet.

2. $V \notin \mathscr{L}$, then tokens are left behind: Suppose $V \in \mathscr{L}$ and that we assign a copy of an edge to each of its endpoint. Therefore,

$$\begin{aligned} 2\chi(E(V)) &= \sum_{v\in V} \chi(\delta(v)) \\ &= \sum_{v\in T} \chi(\delta(v)) + \sum_{v\notin T} \chi(\delta(v)) \\ &= \sum_{v\in T} \chi(\delta(v)) + \sum_{v\in V\setminus T} \sum_{e\in\delta(v)} \chi(e) \end{aligned}$$

In the next lecture, we will see why linear independence fails the second case. We will also finish the proof then.

4 Summary

In this lecture we introduced the idea of local token counting argument and applied it to the subtour-LP for MST and the degree-bounded MST problems. We saw a $(1, B_v+2)$-approximation for degree-bounded MST by [Goe06], which was later improved to $(1, B_v+1)$-approximation by [SL15].

References

[Goe06] Michel Goemans. Minimum bounded degree spanning trees. *2006 47th Annual IEEE Symposium on Foundations of Computer Science (FOCS06)*, 2006.

[SL15] Mohit Singh and Lap Chi Lau. Approximating minimum bounded degree spanning trees to within one of optimal. *J. ACM*, 62(1):1:1–1:19, March 2015.