COMPSCI 632: Approximation Algorithms	November 8, 2017
Lecture #20	
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1 Overview

Today, we continue to discuss about metric embeddings technique. Specifically, we apply metric embeddings technique to solve the sparsest cut problem.

2 Embedding to l_p norm

l_p norm of a vector $\boldsymbol{x}$ is defined by

$$\|\boldsymbol{x}\|_p = (\sum_{i=1}^{n} \boldsymbol{x}_i^p)^{1/p}.$$

In this section, we discuss embedding from any metric space to $\mathbb{R}^n$ with l_p norm.

2.1 l_∞ norm

Let's first look at l_∞ norm, where

$$\|\boldsymbol{x}\|_\infty = \lim_{p\to\infty} (\sum_{i=1}^{n} \boldsymbol{x}_i^p)^{1/p} = \max_i \boldsymbol{x}_i.$$

We say an embedding is *isometric* if the distortion of the embedding is 1.

Theorem 1. *l_∞ norm is universal, i.e., given any metric space, there exists an isometric embedding to $\mathbb{R}^n$ (n could be arbitrary) with l_∞ norm.*

Proof. Assume there are n points in the original metric space $\mathscr{M}$, labelled from 1 to k. Let $d(i,j)$ be the distance between the i-th point and j-th point in the metric space $\mathscr{M}$. Define the embedding function $f : [n] \to \mathbb{R}^n$ to be

$$[f(i)]_k = d(i,k)$$

Then, with l_∞ norm, we have

$$\|f(i) - f(j)\|_\infty = \max_k |f_k(i) - f_k(j)| = \max_k |d(i,k) - d(j,k)| \le d(i,j)$$

(Triangular inequality is applied in the last inequality.) Moreover, when we choose the index k to be either i or j, (say i) we have

$$|f_i(i) - f_i(j)| = |d(i,i) - d(i,j)| = d(i,j)$$

since $d(i,i) = 0$. Therefore, we can conclude that

$$\|f(i) - f(j)\|_\infty = d(i,j)$$

□

Exercise 1. *Show that Euclidean (l_2) norm is not universal with the following star graph: $V = \{1,2,3\} \cup \{m\}$ with distance $d(i,m) = 1$ for all $i \in \{1,2,3\}$ and $d(i,j) = 2$ for all $i \ne j$ and $i,j \in \{1,2,3\}$.*

2.2 l_1 norm

For l_1 norm, we have the following theorem, which we will use without proving for the sparsest cut problem.

Theorem 2 ([Bou85])**.** *Any metric space on n points can be deterministicly embedded into an l_1 norm space with $O(\log^2 n)$ dimension and distortion $4\log n$.*

3 Sparsest cut

Let ∂S be the collection of edges in the cut $(S, V \setminus S)$:

$$\partial S = \{(i,j) \in E \mid i \in S, j \in V \setminus S\}$$

and denote the capacity of an edge $(i,j) \in E$ be cap_{ij} and the capacity of a cut by

$$\text{cap}(\partial S) = \sum_{(i,j)\in\partial S} \text{cap}_{ij}$$

Consider a graph $G = (V,E)$. The sparsity of a cut $(S, V \setminus S)$ equals

$$\psi(S) = \frac{\text{cap}(\partial S)}{\min(|S|, |V \setminus S|)}$$

In the sparsest cut problem, the objective is to find a cut with minimum sparsity:

$$\phi(G) = \min_{S\subset V} \psi(S)$$

3.1 Relate to Flux

The flux of a cut G is defined by

$$\text{flux}(G) = \min_{S\subset V} \frac{\text{cap}(\partial \text{S})}{|S| \cdot |V \setminus S|}$$

Notice that for each choice of $S \subset V$,

$$\frac{\phi}{\text{flux}} = \frac{|S| \cdot |V \setminus S|}{n \cdot \min(|S|, |V \setminus S|)} = \frac{1}{n} \cdot \max(|S|, |V \setminus S|) \in [1/2, 1]$$

Therefore, if we can get an α approximation for flux, we can obtain an 2α approximation of ϕ.

3.2 Demand

Let's rewrite the flux function,

$$\text{flux}(G) = \min_{S\subset V} \frac{\sum_{(i,j)\in\partial S} \text{cap}_{ij}}{\sum_{(i,j)\in S\times(V\setminus S)} 1}$$

We can view the constant 1 in the above formula as *demand*, which indicates that for each pair $(i,j) \in S \times V \setminus S$, we need to push an one-unit flow from i to j. The algorithm we are going to present can work if we replace the constant 1 by a demand function: $\text{dem} : V \times V \to \mathbb{R}$. The objective function becomes,

$$f(G) = \min_{S\subset V} \frac{\sum_{(i,j)\in\partial S} \text{cap}_{ij}}{\sum_{(i,j)\in S\times(V\setminus S)} \text{dem}_{ij}}$$

3.3 Cut Metric

Elementary Cut Metric. An elementary cut metric is a metric defined by a cut $(S, V \setminus S)$ such that the distance d_{ij} between i and j is 1 if and only if i and j are separated by the cut; otherwise, $d_{ij} = 0$:

$$d_{ij} = 1 \text{ iff } |\{i,j\} \cap S| = 1$$

With elementary cut metric, we can change the search space from finding a cut to finding an elementary cut metric and rewrite the objective function as follows:

$$f(G) = \min_{d} \frac{\sum_{(i,j)\in\partial S} \text{cap}_{ij} \cdot d_{ij}}{\sum_{(i,j)\in S\times(V\setminus S)} \text{dem}_{ij} \cdot d_{ij}}$$

(general) Cut Metric. A general cut metric is a linear combination of some elementary cut metrics

$$d_{ij} = \sum_{S:|\{i,j\}\cap S|=1} y_S$$

3.4 LP formulation

First notice that the value of the objective is the same up to scaling. Therefore, without loss of generality, we can constrain that

$$\sum_{(i,j)\in S\times(V\setminus S)} \text{dem}_{ij} \cdot d_{ij} = 1$$

and turn the objective to be

$$\min \sum_{(i,j)\in\partial S} \text{cap}_{ij} \cdot d_{ij}.$$

The remaining constraints make sure that d comes from a metric space. The entire program is as follows:

$$\begin{array}{lll}
\min & \sum_{|\{i,j\}\cap S|=1} \text{cap}_{ij} \cdot d_{ij} & \\
s.t & \sum_{|\{i,j\}\cap S|=1} \text{dem}_{ij} \cdot d_{ij} = 1 & \\
 & d_{ii} = 0 & \forall i \\
 & d_{ij} = d_{ji} & \forall i,j \\
 & d_{ij} + d_{jk} \leq dik & \forall i,j,k \\
 & d \text{ is an elementary cut metric} &
\end{array}$$

We change ∂S to $S \times (V \setminus S)$ in the above LP by letting $\text{cap}_{ij} = 0$ if $(i,j) \in S \times (V \setminus S)$ but $(i,j) \notin E$ and also rewrite $(i,j) \in S \times (V \setminus S)$ by $|\{i,j\} \cap S| = 1$ for simlicity. Finally, in order to obtain an LP, we drop the last constraint so that we compute the best metric instead of the best elementary cut metric.

3.5 Analysis

Given the solution metric d^* from LP, we first apply Theorem 2 to embed it to metric d^{l_1} in $\mathbb{R}^{\log^2 n}$ with l_1 norm. In the next step, we turn the d^{l_1} to a (general) cut metric d^{gc}, which we will show that this embedding is isometric. Finally, we extract an elementary cut metric d^{ec} from d^{gc} to obtain our solution.

Claim 1. *Embedding d^{l_1} to d^{gc} is isometric.*

Proof. Let's first consider the case when d^{l_1} is in one dimension. Therefore, all the points are located on a line. Without loss of generality, assume these points to be $x_1 < x_2 < x_n$. For each $1 \le i < n$, we define a cut between x_i and x_{i+1} such that $S_i = \{1, \cdots, i\}$ and let $y_{S_i} = x_{i+1} - x_i$. Then, for any pair (i, j), we have

$$d_{ij}^{l_1} = \sum_{i \le k < j} y_{S_i} = \sum_{i \le k < j} x_{k+1} - x_k = x_j - x_i.$$

Therefore, embedding d^{l_1} to d^{gc} is isometric when d^{l_1} is in one dimension. Notice that l_1 distance between $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$ is

$$\|\boldsymbol{x}_i - \boldsymbol{x}_j\|_1 = \sum_k |(\boldsymbol{x}_i)_k - (\boldsymbol{x}_j)_k|$$

Thus, we can apply the argument for one dimension case to each dimension separately to show that the embedding is isometric even when d^{l_1} is in a higher dimension space. □

Recall that after applying Theorem 2, d^{l_1} is in $O(\log^2 n)$ dimension. According to the proof of Claim 1, the cut metric d^{gc} is a linear combination of $O(n \log^2 n)$ elementary cut metrics (denote the set of these cut by EC). Therefore, we can find the elementary cut metric d^{ec} with minimum objective value in poly time. It remains to show that the elementary cut metric we obtain is a good solution.

$$\begin{aligned}
\min_{S \in EC} \frac{\sum_{|\{i,j\} \cap S|=1} \mathrm{cap}_{ij}}{\sum_{|\{i,j\} \cap S|=1} \mathrm{dem}_{ij}} &= \min_{S \in EC} \frac{y_S \cdot \sum_{|\{i,j\} \cap S|=1} \mathrm{cap}_{ij}}{y_S \cdot \sum_{|\{i,j\} \cap S|=1} \mathrm{dem}_{ij}} \le \frac{\sum_{S \in EC} y_S \cdot \sum_{|\{i,j\} \cap S|=1} \mathrm{cap}_{ij}}{\sum_{S \in EC} y_S \cdot \sum_{|\{i,j\} \cap S|=1} \mathrm{dem}_{ij}} \\
&= \frac{\sum_{(i,j)} \mathrm{cap}_{ij} \cdot \sum_{S \in EC, |\{i,j\} \cap S|=1} y_S}{\sum_{(i,j)} \mathrm{dem}_{ij} \sum_{S \in EC, |\{i,j\} \cap S|=1} y_S} = \frac{\sum_{(i,j)} \mathrm{cap}_{ij} \cdot d_{ij}^{l_1}}{\sum_{(i,j)} \mathrm{dem}_{ij} \cdot d_{ij}^{l_1}} \\
&\le \frac{\sum_{(i,j)} \mathrm{cap}_{ij} \cdot 4 \log n \cdot d_{ij}^*}{\sum_{(i,j)} \mathrm{dem}_{ij} \cdot d_{ij}^*} \le 4 \log n LP \le 4 \log n OPT
\end{aligned}$$

The first step is just to multiply y_S on both the denominator and the nominator, which still keeps the function value the same. The second step applies the following claim

Claim 2. *If $a_i > 0$ and $b_i > 0$ for all i, we have*

$$\min_i \frac{a_i}{b_i} \le \frac{\sum_i a_i}{\sum_i b_i}.$$

The third step is a change of order of summation while the fourth step uses the fact that $\sum_{S \in EC, |\{i,j\} \cap S|=1} y_S = d_{ij}^{gc}$ and Claim 1. In the fourth step, we apply Theorem 2 with $d_{ij}^* \le d_{ij}^{l_1} \le 4 \log n \cdot d_{ij}^*$.

4 Summary

In this lecture, we discuss embedding to l_p norm and use the embedding to l_1 norm to design an approximation algorithm for the sparsest cut problem [LLR95].

References

[Bou85] Jean Bourgain. On lipschitz embedding of finite metric spaces in hilbert space. *Israel Journal of Mathematics*, 52(1):46–52, 1985.

[LLR95] Nathan Linial, Eran London, and Yuri Rabinovich. The geometry of graphs and some of its algorithmic applications. *Combinatorica*, 15(2):215–245, 1995.