COMPSCI 632: Approximation Algorithms	November 13, 2017
Lecture 21	
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1 Overview

Today we continue our discussion on the sparsest cut problem. After a brief review of our LP relaxation method, we start showing why this method has its approximation limit. In the end, we start discussing on a new relaxation method.

2 Brief Review

Definition 1. For any $S \subseteq V$, $\partial S := \{(i,j) | i \in S, j \notin S\}$.

Definition 2. Define the *capacity* of an edge set as the sum of the capacities of edges in the set, i.e. $\text{cap}(E') := \sum_{(i,j)\in E'} c_{ij}$.

Definition 3. Define the *sparsity* of a cut $(S, V\backslash S)$ as:

$$\psi(S) := \frac{\text{cap}(\partial S)}{\min(|S|, |V\backslash S|)}.$$

Also $\phi(G) := \min_{S\subseteq V} \psi(S)$.

Definition 4. Define the *flux* of a cut $(S, V\backslash S)$ as:

$$\text{flux}(S) := \frac{\text{cap}(\partial S)}{|S|\cdot|V\backslash S|}.$$

From the previous lecture we know if we can get an α-approximation of the minimum-flux cut problem, we get a 2α-approximation of the minimum-sparsity cut problem (and vice versa). We will focus on the minimum-flux cut problem from now on.

Definition 5. An *elementary cut metric* given by the cut $(S, V\backslash S)$ is defined as:

$$l_{ij} := |\{i,j\} \cap S| \text{ mod } 2.$$

Definition 6. A *cut metric* is the weighted sum of some elementary cut metrics.

The metric view of the problem is:

$$\min_{\text{cut metric } l} \frac{\sum_{i\in V, j\in V} c_{ij} l_{ij}}{\sum_{i\in V, j\in V} d_{ij} l_{ij}},$$

where the *demand* $d_{ij} \equiv 1$. Note that in later discussion, the notion seems to be generalized. In particular, it is possible that $d_{ij} = 0$ and $c_{ij} \neq 0$.

We can write a linear program for the problem:

$$\begin{array}{lll} \text{minimize} & \sum_{i,j} c_{ij} l_{ij} & \\ \text{subject to} & \sum_{ij} d_{ij} l_{ij} \geq 1, & \\ & l_{ij} + l_{jk} \geq l_{ik} & \forall i,j,k, \\ & l_{ij} \geq 0 & \forall i,j. \end{array}$$

Last time we discussed how this can give a $O(\log n)$-approximation by using Bourgain's theorem [Bou85]. Now we are going to show it is the best possible using the LP formulation, by showing its integrality gap.

3 Duality View

Consider the maximum concurrent flow problem, which can be formulated by the following LP:

$$\begin{array}{lll} \text{maximize} & \lambda & \\ \text{subject to} & \sum_{p \in \mathscr{P}_{ij}} f_p \geq \lambda \cdot d_{ij} & \forall i,j, \\ & \sum_{p:(i,j) \in p} f_p \leq c_{ij} & \forall i,j, \\ & f_p \geq 0 & \forall p. \end{array}$$

The dual of it is:

$$\begin{array}{lll} \text{minimize} & \sum_{i,j} c_{ij} \beta_{ij} & \\ \text{subject to} & \sum_{i,j} \alpha_{ij} d_{ij} \geq 1, & \\ & \left(\sum_{(i,j) \in p} \beta_{ij} \right) - \alpha_{p_s p_t} \geq 0 & \forall p, \\ & \alpha_{ij}, \beta_{ij} \geq 0 & \forall i,j. \end{array}$$

In fact, this dual is equivalent to the LP formulation of the sparsest cut problem. To see that, notice α_{ij} is at most the shortest path distance under β_{ij}'s, and has no reason to be smaller.

4 Integrality Gap for the Linear Programming Formulation

Example 1. Consider the graph in Figure 1. Every bold black edge has $c = 1$ and $d = 0$. Every red dotted edge has $c = 0$ and $d = 1$. Then the minimum sparsity is 1 (by cutting only the upper right vertex). However, our LP has a solution of $\frac{3}{4}$, by setting the metric as follows: Every two vertices on the same-hand side (i.e. connected by a red dotted edge) has a distance of 2. Every other pair of vertices has a distance of 1. Therefore we know our LP formulation has an integrality gap of $\frac{4}{3}$.

In fact, we can have better bounds by looking at *expander graphs*.

Definition 7. A *constant-degree expander* is a graph in which

$$\min_{S \in V, |S| \leq \frac{|V|}{2}} \frac{|\partial S|}{|S|} = \Omega(1).$$

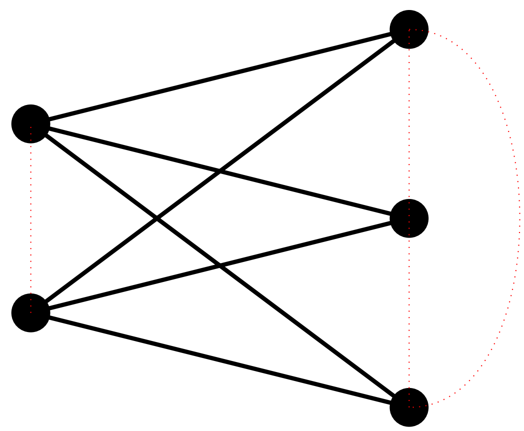

Figure 1: An Example to Show the Integrality Gap

Consider the graph where we assign each vertex to have three random neighbors. It is a constant-degree expander and there are $\Omega(n^2)$ pairs of vertices at a distance of $\Omega(\log n)$ from each other. In such a graph, the actual optimum is only $\Theta\left(\frac{1}{n}\right)$, but our relaxation will result in a solution of $\Theta\left(\frac{1}{n\log n}\right)$. This shows an integrality gap of $\Theta(\log n)$.

5 New Idea

To avoid the $\Theta(\log n)$ integrality gap, we will try the following idea: instead of relaxing l_1-norm to any metric, we relax l_1-norm to l_2^2-metrics. Notice that in order that l_2^2 is a metric, all angles between the points must be *acute* (by the law of cosines). We will elaborate on this idea in the next lecture.

6 Summary

Today we discussed further on the sparsest cut problem. We provided a duality view of our LP relaxation, showed its integrality gap, and started talking about another way to relax the problem. We will continue on this topic in the next lecture.

References

[Bou85] Jean Bourgain. On lipschitz embedding of finite metric spaces in hilbert space. *Israel Journal of Mathematics*, 52(1):46–52, 1985.