COMPSCI 638: Problems

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Due on December 16, 2019

Problem 1. *Answer the following questions on maximum flow algorithms:*

(a) *Given a unit-capacity directed graph with n vertices and m edges, show that $O(n^{2/3})$ blocking flows suffice in finding a maximum flow. (Recall that we proved an upper bound of $O(n)$ blocking flows for arbitrary graphs in class.) Now, use this property to obtain an $O(mn^{2/3})$-time maximum flow algorithm for unit-capacity directed graphs.*

(b) *If every vertex has either in-degree or out-degree at most 1, show a tighter bound of $O(n^{1/2})$ on the number of blocking flows that suffice in finding a maximum flow. Use this tighter bound to obtain an $O(mn^{1/2})$-time algorithm for finding a maximum matching in an undirected bipartite graph.*

Problem 2. *A hypergraph $G = (V, E)$ is a generalization of an undirected graph where V is the set of vertices (as in a graph) and $E \subseteq 2^V$ is a set of hyperedges, each of which is a subset of the vertices (a graph only allows subsets of two vertices each). The* rank *of a hypergraph is the largest cardinality of a hyperedge. Show that:*

(a) *A hypergraph has at most $\binom{n}{2}$ global minimum cuts irrespective of its rank. (Hint: Use a randomized contraction procedure similar to that in graphs, but contract hyperedges in a non-uniform manner.)*

(b) *A hypergraph of rank r can have an exponential (in r) 2-minimum cuts, where a 2-minimum cut has at most twice as many hyperedges as a global minimum cut. (Hint: Give an example of a family of hypergraphs with this property.)*

Problem 3. *Answer the following questions on spectral graph theory:*

(a) *Show that the number of connected components of an undirected graph is equal to the multiplicity of 0 as an eigenvalue.*

(b) *Suppose for an undirected, regular graph $G = (V, E)$ of degree d, we define:*

$$\lambda_G := \min_{\mathbf{x} \perp \mathbf{1}} \frac{\sum_{(i,j)\in E}(x_i - x_j)^2}{d \cdot \sum_{i \in V} x_i^2}, \text{ where } \mathbf{1} \text{ is the all-ones vector in } |V| \text{ dimensions, and}$$

$$\phi_G := \min_{S \subset V : |S| \leq \frac{|V|}{2}} \frac{|E(S, \bar{S})|}{d \cdot |S|}, \text{ where } |E(S, \bar{S})| \text{ is the size of the cut } (S, \bar{S}).$$

Show that: $\phi_G \geq \frac{\lambda_G}{2}$. (Hint: If you are unable to prove the above statement, try proving $\phi_G \geq c \cdot \lambda_G$ for any constant c for partial credit.)

Problem 4. *A* cut tree *is a (weighted) tree $T_G = (V, E_T)$ defined on the vertices of an undirected graph $G = (V, E)$ such that for any pair of vertices $s, t \in V$, the value of the $s - t$ min-cut in T is equal to the value of the $s - t$ min-cut in G. If G is unweighted, show the following:*

(a) *The sum of weights of the edges of the cut tree T_G is at most $2|E|$.*

(b) *If d is the minimum degree of a vertex in G, then the number of $d/2$-edge-connected components in G is at most $\frac{3|V|}{4}$. (Hint: Use cut trees in the proof.)*