Methods for Transforming Grammars

We will consider CFL without $\lambda$. It would be easy to add $\lambda$ to any grammar by adding a new start symbol $S_0$,

$$S_0 \rightarrow S \mid \lambda$$
Theorem (Substitution) Let G be a CFG. Suppose G contains

\[ A \rightarrow x_1Bx_2 \]

where A and B are different variables, and B has the productions

\[ B \rightarrow y_1 | y_2 | \ldots | y_n \]

Then can construct G’ from G by deleting

\[ A \rightarrow x_1Bx_2 \]

from P and adding to it

\[ A \rightarrow x_1y_1x_2 | x_1y_2x_2 | \ldots | x_1y_nx_2 \]

Then, \( L(G) = L(G') \).
Example:

\[ S \rightarrow aBa \quad \text{becomes} \]
\[ B \rightarrow aS \mid a \]

Definition: A production of the form \( A \rightarrow Ax, \ A \in V, \ x \in (V \cup T)^* \) is left recursive.
Example Previous expression grammar was left recursive.

\[
\begin{align*}
E & \rightarrow E + T \mid T \\
T & \rightarrow T * F \mid F \\
F & \rightarrow I \mid (E) \\
I & \rightarrow a \mid b
\end{align*}
\]

Derivation of \(a + b + a + a\) is:

\[
\begin{align*}
E \Rightarrow & \ E + T \Rightarrow E + T + T \Rightarrow E + T + T + T \\
\Rightarrow & \ a + T + T + T
\end{align*}
\]
Theorem (Removing Left recursion)
Let $G=(V,T,S,P)$ be a CFG. Divide productions for variable $A$ into left-recursive and non left-recursive productions:

\[
A \rightarrow A.x_1 \mid A.x_2 \mid \ldots \mid A.x_n \\
A \rightarrow y_1|y_2|\ldots|y_m
\]

where $x_i, y_i$ are in $(V \cup T)^*$.

Then $G’=(V\cup\{Z\}, T, S, P’)$ and $P’$ replaces rules of form above by

\[
A \rightarrow y_i|y_iZ, \ i=1,2,\ldots,m \\
Z \rightarrow x_i|x_iZ, \ i=1,2,\ldots,n
\]
Example:

\[ E \rightarrow E + T | T \]  becomes

\[ T \rightarrow T * F | F \]  becomes

Now, Derivation of \( a + b + a + a \) is:
Useless productions

\[ S \rightarrow aB \mid bA \]
\[ A \rightarrow aA \]
\[ B \rightarrow Sa \mid a \]
\[ C \rightarrow cBc \mid a \]

Theorem (useless productions) Let \( G \) be a CFG. Then \( \exists \ G' \) that does not contain any useless variables or productions s.t. \( L(G) = L(G') \).
To Remove Useless Productions:
Let $G=(V,T,S,P)$.

I. Compute $V_1=\{\text{Variables that can derive strings of terminals}\}$

1. $V_1=\emptyset$

2. Repeat until no more variables added

   ● For every $A \in V$ with $A \rightarrow x_1x_2 \ldots x_n$, $x_i \in (T^* \cup V_1)$, add $A$ to $V_1$

3. $P_1 = \text{all productions in } P \text{ with symbols in } (V_1 \cup T)^*$

Then $G_1=(V_1,T,S,P_1)$ has no variables that can’t derive strings.
II. Draw Variable Dependency Graph

For $A \rightarrow xBy$, draw $A \rightarrow B$.

Remove productions for $V$ if there is no path from $S$ to $V$ in the dependency graph. Resulting Grammar $G'$ is s.t. $L(G) = L(G')$ and $G'$ has no useless productions.
Example:

\[
\begin{align*}
S & \rightarrow aB \ | \ bA \\
A & \rightarrow aA \\
B & \rightarrow Sa \ | \ a \\
C & \rightarrow cBc \ | \ a
\end{align*}
\]
Theorem (remove $\lambda$ productions) Let $G$ be a CFG with $\lambda$ not in $L(G)$. Then $\exists$ a CFG $G'$ having no $\lambda$-productions s.t. $L(G) = L(G')$.

To Remove $\lambda$-productions

1. Let $V_n = \{A \mid \exists \text{ production } A \rightarrow \lambda \}$

2. Repeat until no more additions
   • if $B \rightarrow A_1 A_2 \ldots A_m$ and $A_i \in V_n$ for all $i$, then put $B$ in $V_n$

3. Construct $G'$ with productions $P'$ s.t.
   • If $A \rightarrow x_1 x_2 \ldots x_m \in P$, $m \geq 1$, then put all productions formed when $x_j$ is replaced by $\lambda$ (for all $x_j \in V_n$) s.t. $|\text{rhs}| \geq 1$ into $P'$. 
Example:

\[
S \rightarrow Ab \\
A \rightarrow BC \mid Aa \\
B \rightarrow b \mid \lambda \\
C \rightarrow cC \mid \lambda
\]
Definition Unit Production

\[ A \rightarrow B \]

where \(A, B \in V\).

Consider removing unit productions:

Suppose we have
\[ A \rightarrow B \] becomes
\[ B \rightarrow a \mid ab \]

But what if we have
\[ A \rightarrow B \] becomes
\[ B \rightarrow C \]
\[ C \rightarrow A \]
Theorem (Remove unit productions)
Let \( G=(V,T,S,P) \) be a CFG without \( \lambda \)-productions. Then \( \exists \) CFG \( G'=(V',T',S,P') \) that does not have any unit-productions and \( L(G)=L(G') \).

To Remove Unit Productions:

1. Find for each \( A \), all \( B \) s.t. \( A \Rightarrow B \)
   (Draw a dependency graph)
2. Construct \( G'=(V',T',S,P') \) by
   
   (a) Put all non-unit productions in \( P' \)
   
   (b) For all \( A \Rightarrow^* B \) s.t. \( B \rightarrow y_1|y_2| \ldots y_n \in P' \), put \( A \rightarrow y_1|y_2| \ldots y_n \in P' \)
Example:

S $\rightarrow$ AB
A $\rightarrow$ B
B $\rightarrow$ C | Bb
C $\rightarrow$ A | c
Theorem Let \( L \) be a CFL that does not contain \( \lambda \). Then \( \exists \) a CFG for \( L \) that does not have any useless productions, \( \lambda \)-productions, or unit-productions.

Proof

1. Remove \( \lambda \)-productions
2. Remove unit-productions
3. Remove useless productions

Note order is very important. Removing \( \lambda \)-productions can create unit-productions! QED.
Definition: A CFG is in Chomsky Normal Form (CNF) if all productions are of the form

\[ A \rightarrow BC \quad \text{or} \quad A \rightarrow a \]

where \( A, B, C \in V \) and \( a \in T \).

Theorem: Any CFG \( G \) with \( \lambda \) not in \( L(G) \) has an equivalent grammar in CNF.

Proof:

1. Remove \( \lambda \)-productions, unit productions, and useless productions.

2. For every rhs of length > 1, replace each terminal \( x_i \) by a new variable \( C_j \) and add the production \( C_j \rightarrow x_i \).

3. Replace every rhs of length > 2 by a series of productions, each with rhs of length 2. QED.
Example:

\[ S \rightarrow CBcd \]
\[ B \rightarrow b \]
\[ C \rightarrow Cc \mid e \]
Definition: A CFG is in Greibach normal form (GNF) if all productions have the form

\[ A \rightarrow ax \]

where \( a \in T \) and \( x \in V^* \)

Theorem For every CFG \( G \) with \( \lambda \) not in \( L(G) \), \( \exists \) a grammar in GNF.

Proof:

1. Rewrite grammar in CNF.
2. Relabel Variables \( A_1, A_2, \ldots A_n \)
3. Eliminate left recursion and use substitution to get all productions into the form:

\[ A_i \rightarrow A_jx_j, \ j > i \]
\[ Z_i \rightarrow A_jx_j, \ j \leq n \]
\[ A_i \rightarrow ax_i \]

where \( a \in T, \ x_i \in V^*, \) and \( Z_i \) are new variables introduced for left recursion.

4. All productions with \( A_n \) are in the correct form, \( A_n \rightarrow ax_n. \) Use these productions as substitutions to get \( A_{n-1} \) productions in the correct form. Repeat with \( A_{n-2}, A_{n-3}, \) etc until all productions are in the correct form.