A DFA = (Q, Σ, δ, q_0, F)
Modify DFA by adding a stack. New machine is called Pushdown Automata (PDA).
Definition: Nondeterministic PDA (NPDA) is defined by

\[ M = (Q, \Sigma, \Gamma, \delta, q_0, z, F) \]

where
- \( Q \) is finite set of states
- \( \Sigma \) is tape (input) alphabet
- \( \Gamma \) is stack alphabet
- \( q_0 \) is initial state
- \( z \) is start stack symbol (bottom of stack)
- \( F \subseteq Q \) is set of final states.
- \( \delta: Q \times (\Sigma \cup \{\lambda\}) \times \Gamma \rightarrow \text{finite subsets of } Q \times \Gamma^* \)
Example of transitions

\[ \delta(q_1, a, b) = \{(q_3, b), (q_4, ab), (q_6, \lambda)\} \]

The diagram for the above transitions is:
Instantaneous Description:

\[(q,w,u)\]

Description of a Move:

\[(q_1,aw,bx) \vdash (q_2,w,yx)\]

iff

Definition Let \(M=(Q, \Sigma, \Gamma, \delta, q_0, z, F)\) be a NPDA. \(L(M)=\{w \in \Sigma^* \mid (q_0,w,z) \vdash^* (p,\lambda,u), p \in F, u \in \Gamma^*\}\). The NPDA accepts all strings that start in \(q_0\) and end in a final state.
Example: $L = \{a^n b^n | n \geq 0\}$, $\Sigma = \{a, b\}$, $\Gamma = \{z, a\}$
Another Definition for Language Acceptance

NPDA $M$ accepts $L(M)$ by empty stack:

$$L(M) = \{ w \in \Sigma^* | (q_0, w, z)^* \vdash (p, \lambda, \lambda) \}$$
Example: \( L = \{ww^R | w \in \Sigma^+ \} \), \( \Sigma = \{a, b\} \),
\( \Gamma = \{z, a, b\} \)
Example: $L=\{ww|w \in \Sigma^*\}, \Sigma = \{a, b\}$
Examples for you to try on your own: (solutions are at the end of the handout).

- \( L = \{a^n b^m | m > n, m, n > 0\} \), \( \Sigma = \{a, b\} \), 
  \( \Gamma = \{z, a\} \)

- \( L = \{a^n b^{n+m} c^m | n, m > 0\} \), \( \Sigma = \{a, b, c\} \), 

- \( L = \{a^n b^{2n} | n > 0\} \), \( \Sigma = \{a, b\} \)
Theorem Given NPDA M that accepts by final state, ∃ NPDA M’ that accepts by empty stack s.t. L(M)=L(M’).

• Proof (sketch)
  M=(Q,Σ,Γ,δ,q₀,z,F)
  Construct M’=(Q’,Σ,Γ’,δ’,qₛ,z’,F’)

Theorem Given NPDA $M$ that accepts by empty stack, $\exists$ NPDA $M'$ that accepts by final state.

- Proof: (sketch)
  $M = (Q, \Sigma, \Gamma, \delta, q_0, z, F)$
  Construct $M' = (Q', \Sigma, \Gamma', \delta', q_s, z', F')$
Theorem For any CFL $L$ not containing $\lambda$, $\exists$ an NPDA $M$ s.t. $L=L(M)$.

- Proof (sketch)
  
  Given (\lambda-free) CFL $L$.
  
  $\Rightarrow \exists$ CFG $G$ such that $L=L(G)$.
  
  $\Rightarrow \exists G'$ in GNF, s.t. $L(G)=L(G')$.
  
  $G'=(V,T,S,P)$. All productions in $P$ are of the form:
Example: Let $G'=(V,T,S,P)$, $P =$

\[
S \rightarrow aSA \mid aAA \mid b \\
A \rightarrow bBBB \\
B \rightarrow b
\]
Theorem Given a NPDA M, \( \exists \) a NPDA M’ s.t. all transitions have the form \( \delta(q_i,a,A) = \{c_1, c_2, \ldots, c_n\} \) where

\[
\begin{align*}
  c_i &= (q_j, \lambda) \\
  \text{or} \quad c_i &= (q_j, BC)
\end{align*}
\]

Each move either increases or decreases stack contents by a single symbol.

• Proof (sketch)
Theorem: If $L = L(M)$ for some NPDA $M$, then $L$ is a CFL.

• Proof: Given NPDA $M$.

First, construct an equivalent NPDA $M$ that will be easier to work with. Construct $M'$ such that

1. accepts if stack is empty
2. each move increases or decreases stack content by a single symbol. (can only push 2 variables or no variables with each transition)

$M' = (Q, \Sigma, \Gamma, \delta, q_0, z, F)$

Construct $G = (V, \Sigma, S, P)$ where

$V = \{(q_i c q_j) | q_i, q_j \in Q, c \in \Gamma\}$

Goal: $(q_0 z q_f)$ which will be the start symbol in the grammar.
Example:

\[ L(M) = \{aa^*b\}, \ M = (Q, \Sigma, \Gamma, \delta, q_0, z, F), \]
\[ Q = \{q_0, q_1, q_2, q_3\}, \]
\[ \Sigma = \{a, b\}, \Gamma = \{A, z\}, F = \{\} . \]
Construct the grammar $G=(V,T,S,P)$,

$V=\{(q_0Aq_0), (q_0zq_0), (q_0Aq_1), (q_0zq_1), \ldots\}$

$T=\Sigma$

$S=(q_0zq_2)$

$P=$
From transition 1 \((q_0Aq_1) \rightarrow b\)

From transition 2 \((q_1zq_2) \rightarrow \lambda\)

From transition 3 \((q_0Aq_3) \rightarrow a\)

From transition 4 \((q_0zq_0) \rightarrow a(q_0Aq_0)(q_0zq_0)\) |
\(a(q_0Aq_1)(q_1zq_0)\) |
\(a(q_0Aq_2)(q_2zq_0)\) |
\(a(q_0Aq_3)(q_3zq_0)\)

\((q_0zq_1) \rightarrow a(q_0Aq_0)(q_0zq_1)\) |
\(a(q_0Aq_1)(q_1zq_1)\) |
\(a(q_0Aq_2)(q_2zq_1)\) |
\(a(q_0Aq_3)(q_3zq_1)\)

\((q_0zq_2) \rightarrow a(q_0Aq_0)(q_0zq_2)\) |
\(a(q_0Aq_1)(q_1zq_2)\) |
\(a(q_0Aq_2)(q_2zq_2)\) |
\(a(q_0Aq_3)(q_3zq_2)\)

\((q_0zq_3) \rightarrow a(q_0Aq_0)(q_0zq_3)\) |
\(a(q_0Aq_1)(q_1zq_3)\) |
\(a(q_0Aq_2)(q_2zq_3)\) |
\(a(q_0Aq_3)(q_3zq_3)\)
From transition 5 \((q_3zq_0) \rightarrow (q_0Aq_0)(q_0zq_0)\)  
\((q_0Aq_1)(q_1zq_0)\)  
\((q_0Aq_2)(q_2zq_0)\)  
\((q_0Aq_3)(q_3zq_0)\)  
\((q_3zq_1) \rightarrow (q_0Aq_0)(q_0zq_1)\)  
\((q_0Aq_1)(q_1zq_1)\)  
\((q_0Aq_2)(q_2zq_1)\)  
\((q_0Aq_3)(q_3zq_1)\)  
\((q_3zq_2) \rightarrow (q_0Aq_0)(q_0zq_2)\)  
\((q_0Aq_1)(q_1zq_2)\)  
\((q_0Aq_2)(q_2zq_2)\)  
\((q_0Aq_3)(q_3zq_2)\)  
\((q_3zq_3) \rightarrow (q_0Aq_0)(q_0zq_3)\)  
\((q_0Aq_1)(q_1zq_3)\)  
\((q_0Aq_2)(q_2zq_3)\)  
\((q_0Aq_3)(q_3zq_3)\)
Recognizing \textit{aaab} in M:

\((q_0, aaab, z) \Rightarrow (q_0, aab, Az) \Rightarrow (q_3, ab, z) \Rightarrow (q_0, ab, Az) \Rightarrow (q_3, b, z) \Rightarrow (q_0, b, Az) \Rightarrow (q_1, \lambda, z) \Rightarrow (q_2, \lambda, \lambda)\)

Derivation of string \textit{aaab} in G:

\((q_0zq_2) \Rightarrow a(q_0Aq_3)(q_3zq_2) \Rightarrow aa(q_3zq_2) \Rightarrow aa(q_0Aq_3)(q_3zq_2) \Rightarrow aaa(q_3zq_2) \Rightarrow aaaa(q_0Aq_1)(q_1zq_2) \Rightarrow aaab(q_1zq_2) \Rightarrow aaab\)
Definition: A PDA

\[ M = (Q, \Sigma, \Gamma, \delta, q_0, z, F) \] is deterministic if for every \( q \in Q, a \in \Sigma \cup \{ \lambda \}, b \in \Gamma \):

1. \( \delta(q, a, b) \) contains at most 1 element
2. if \( \delta(q, \lambda, b) \neq \emptyset \) then \( \delta(q, c, b) = \emptyset \) for all \( c \in \Sigma \)

Definition: \( L \) is DCFL iff \( \exists \) DPDA M s.t. \( L = L(M) \).

Examples:

1. Previous pda for \( \{ a^n b^n | n \geq 0 \} \) is deterministic.

2. Previous pda for \( \{ a^n b^m c^{n+m} | n, m > 0 \} \) is deterministic.

3. Previous pda for \( \{ w w^R | w \in \Sigma^+ \}, \Sigma = \{ a, b \} \) is nondeterministic.

Note: There are CFL’s that are not deterministic.
L = \{a^n b^n \mid n \geq 1\} \cup \{a^n b^{2n} \mid n \geq 1\} is a CFL and not a DCFL.

- **Proof:**
  
  \begin{align*}
  L &= \{a^n b^n : n \geq 1\} \cup \{a^n b^{2n} : n \geq 1\} \\
  \text{It is easy to construct a NPDA for} \\
  \{a^n b^n : n \geq 1\} \text{ and a NPDA for} \\
  \{a^n b^{2n} : n \geq 1\}. \text{ These two can be} \\
  \text{joined together by a new start state} \\
  \text{and } \lambda\text{-transitions to create a NPDA} \\
  \text{for } L. \text{ Thus, } L \text{ is CFL.} \\
  \text{Now show } L \text{ is not a DCFL.} \\
  \text{Assume that there is a} \\
  \text{deterministic PDA } M \text{ such that} \\
  L = L(M). \text{ We will construct a PDA} \\
  \text{that recognizes a language that is} \\
  \text{not a CFL and derive a} \\
  \text{contradiction.} \\
  \text{Construct a PDA } M' \text{ as follows:} \\
  \end{align*}

1. Create two copies of } M: M_1 \text{ and } M_2. \text{ The same state in } M_1 \text{ and } M_2
are called cousins.

2. Remove accept status from accept states in \( M_1 \), remove initial status from initial state in \( M_2 \). In our new PDA, we will start in \( M_1 \) and accept in \( M_2 \).

3. Outgoing arcs from old accept states in \( M_1 \), change to end up in the cousin of its destination in \( M_2 \). This joins \( M_1 \) and \( M_2 \) into one PDA. There must be an outgoing arc since you must recognize both \( a^n b^n \) and \( a^n b^{2n} \). After reading \( n \) \( b \)'s, must accept if no more \( b \)'s and continue if there are more \( b \)'s.

4. Modify all transitions that read a \( b \) and have their destinations in \( M_2 \) to read a \( c \).

This is the construction of our new PDA.
When we read $a^n b^n$ and end up in an old accept state in $M_1$, then we will transfer to $M_2$ and read the rest of $a^n b^{2n}$. Only the $b$’s in $M_2$ have been replaced by $c$’s, so the new machine accepts $a^n b^n c^n$.

The language accepted by our new PDA is $a^n b^n c^n$. But this is not a CFL. Contradiction! Thus there is no deterministic PDA $M$ such that $L(M) = L$. Q.E.D.
Example: \( L = \{a^nb^m | m > n, m, n > 0 \} \), \( \Sigma = \{a, b\} \), \( \Gamma = \{z, a\} \)

Example: \( L = \{a^n b^{n+m} c^m | n, m > 0 \} \), \( \Sigma = \{a, b, c\} \),

Example: \( L = \{a^n b^{2n} | n > 0 \} \), \( \Sigma = \{a, b\} \)