Section: Properties of Regular Languages

Example

$L = \{a^nba^n \mid n > 0\}$

Closure Properties

A set is closed over an operation if

$L_1, L_2 \in \text{class}$
$L_1 \text{ op } L_2 = L_3$
$\Rightarrow L_3 \in \text{class}$
$L_1 = \{ x \mid x \text{ is a positive even integer} \}$

$L$ is closed under

- addition?
- multiplication?
- subtraction?
- division?

Closure of Regular Languages

Theorem 4.1 If $L_1$ and $L_2$ are regular languages, then

$$L_1 \cup L_2$$
$$L_1 \cap L_2$$
$$L_1 L_2$$
$$\overline{L_1}$$
$$L_1^*$$

are regular languages.
Proof (sketch)

$L_1$ and $L_2$ are regular languages

$\Rightarrow \exists$ reg. expr. $r_1$ and $r_2$ s.t.

$L_1 = L(r_1)$ and $L_2 = L(r_2)$

$r_1 + r_2$ is r.e. denoting $L_1 \cup L_2$

$\Rightarrow$ closed under union

$r_1 r_2$ is r.e. denoting $L_1 L_2$

$\Rightarrow$ closed under concatenation

$r_1^*$ is r.e. denoting $L_1^*$

$\Rightarrow$ closed under star-closure
complementation:
  \( L_1 \) is reg. lang.
  \( \Rightarrow \exists \) DFA \( M \) s.t. \( L_1 = L(M) \)

Construct \( M' \) s.t.
  final states in \( M \) are
    nonfinal states in \( M' \)
  nonfinal states in \( M \) are
    final states in \( M' \)
  \( \Rightarrow \) closed under complementation
intersection:

$L_1$ and $L_2$ are reg. lang.

$\Rightarrow \exists$ DFA $M_1$ and $M_2$ s.t.

$L_1 = L(M_1)$ and $L_2 = L(M_2)$

$M_1 = (Q, \Sigma, \delta_1, q_0, F_1)$

$M_2 = (P, \Sigma, \delta_2, p_0, F_2)$

Construct $M' = (Q', \Sigma, \delta', (q_0, p_0), F')$

$Q' = (Q \times P)$

$\delta'$:

$\delta'((q_i, p_j), a) = (q_k, p_l)$ if

$w \in L(M') \iff w \in L_1 \cap L_2$

$\Rightarrow$ closed under intersection
Example:

Regular languages are closed under

reversal \( L^R \)
difference \( L_1 - L_2 \)
right quotient \( L_1 / L_2 \)
homomorphism \( h(L) \)

Right quotient

Def: \( L_1 / L_2 = \{ x \mid xy \in L_1 \text{ for some } y \in L_2 \} \)

Example:

\[
L_1 = \{ a^* b^* \cup b^* a^* \} \\
L_2 = \{ b^n \mid n \text{ is even, } n > 0 \} \\
L_1 / L_2 =
\]
Theorem If $L_1$ and $L_2$ are regular, then $L_1/L_2$ is regular.

Proof (sketch)

$\exists$ DFA $M=(Q,\Sigma,\delta,q_0,F)$ s.t. $L_1 = L(M)$.

Construct DFA $M'=(Q,\Sigma,\delta,q_0,F')$

For each state $i$ do

Make $i$ the start state (representing $L'_i$)

if $L'_i \cap L_2 \neq \emptyset$ then

put $q_i$ in $F'$ in $M'$

QED.
Homomorphism

Def. Let $\Sigma, \Gamma$ be alphabets. A homomorphism is a function

$$h: \Sigma \rightarrow \Gamma^*$$

Example:

$$\Sigma = \{a, b, c\}, \quad \Gamma = \{0, 1\}$$

- $h(a) = 11$
- $h(b) = 00$
- $h(c) = 0$

$h(bc) =$

$h(ab^*) =$
Questions about regular languages:
L is a regular language.

- Given L, \( \Sigma \), \( w \in \Sigma^* \), is \( w \in L \)?

- Is L empty?

- Is L infinite?

- Does \( L_1 = L_2 \)?
Identifying Nonregular Languages

If a language $L$ is finite, is $L$ regular?

If $L$ is infinite, is $L$ regular?

- $L_1 = \{a^n b^m | n > 0, m > 0 \} = aa^* bb^*$
- $L_2 = \{a^n b^n | n > 0 \}$

Prove that $L_2 = \{a^n b^n | n > 0 \}$ is?

- Proof:
Pumping Lemma: Let $L$ be an infinite regular language. \( \exists \) a constant $m > 0$ such that any $w \in L$ with $|w| \geq m$ can be decomposed into three parts as $w = xyz$ with

\[
|xy| \leq m \\
|y| \geq 1 \\
xy^iz \in L \enspace \text{for all } i \geq 0
\]
To Use the Pumping Lemma to prove $L$ is not regular:

- **Proof by Contradiction.**
  Assume $L$ is regular.
  \[ \Rightarrow L \text{ satisfies the pumping lemma.} \]
  Choose a long string $w$ in $L$, $|w| \geq m$.
  Show that there is NO division of $w$ into $xyz$ (must consider all possible divisions) such that $|xy| \leq m$, $|y| \geq 1$ and $xy^iz \in L \ \forall \ i \geq 0$.
  The pumping lemma does not hold. Contradiction!
  \[ \Rightarrow L \text{ is not regular. QED.} \]
Example $L = \{a^n c b^n | n > 0\}$

$L$ is not regular.

- **Proof:**
  
  Assume $L$ is regular.
  
  $\Rightarrow$ the pumping lemma holds.
  
  Choose $w =$
Example \( L = \{ a^n b^{n+s} c^s | n, s > 0 \} \)

\( L \) is not regular.

- **Proof:**
  
  Assume \( L \) is regular.

  \( \Rightarrow \) the pumping lemma holds.

  Choose \( w = \)

  So the partition is:
Example $\Sigma = \{a, b\}$,
$L = \{w \in \Sigma^* \mid n_a(w) > n_b(w)\}$

$L$ is not regular.

- Proof:
  Assume $L$ is regular.
  $\Rightarrow$ the pumping lemma holds.
  Choose $w =$
  So the partition is:
Example $L = \{a^3b^n c^{n-3} | n > 3\}$

$L$ is not regular.
To Use Closure Properties to prove \( L \) is not regular:

- **Proof Outline:**
  Assume \( L \) is regular.
  Apply closure properties to \( L \) and other regular languages, constructing \( L' \) that you know is not regular.
  
  closure properties \( \Rightarrow \) \( L' \) is regular.
  Contradiction!
  
  \( L \) is not regular. QED.

**Example** \( L = \{a^3b^n c^{n-3} | n > 3\} \)

\( L \) is not regular.

- **Proof:** (proof by contradiction)
  Assume \( L \) is regular.
  Define a homomorphism \( h : \Sigma \rightarrow \Sigma^* \)
  \( h(a) = a \quad h(b) = a \quad h(c) = b \)
  
  \( h(L) = \)
Example \( L = \{a^nb^ma^m | m \geq 0, n \geq 0 \} \)

\( L \) is not regular.

- **Proof:** (proof by contradiction)
  Assume \( L \) is regular.
Example: \( L_1 = \{a^n b^n a^n | n > 0 \} \)

\( L_1 \) is not regular.