# CompSci 102 Discrete Mathematics for CS
# Spring 2005 Forbes HW 4

## Due Wednesday, April 4

**Please note the following on your assignment**: (1) all people with whom you collaborated or contacted outside of the instructor and TA, (2) resources you used other than the book and class notes, and (3) how much time you spent working on the assignment.

1. **(5 pts.) Counting**

   Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with 6 members if it must have more women than men?

2. **(5 pts.) Pascal's Identity**

   Using Pascal's Identity, show

$$\binom{n+1}{m} = \binom{n}{m} + \binom{n-1}{m-1} + \binom{n-2}{m-2} + \ldots + \binom{n-m}{0}$$

3. **(10 pts.) Game**

   The Chevalier de Méré proposes two games to you.

   (a) He bets that at least one 6 would appear during a total of four rolls of one die

   (b) He bets that he would roll a double 6, once in 24 rolls of two dice

   In each game, if he is successful, you pay him \$10, if he is unsuccessful, he pays you \$10. Should you take either bet? Which game is better for you?

4. **(15 pts.) Multiple choice tests**

   Answer each of the following *using the definition of expectation.* Be sure to define all your random variables precisely.

   (a) A teacher is setting a multiple choice test. Each question has five possible responses, exactly one of which is correct. One point is awarded for each correct answer, and $-b$ points for each incorrect answer. The teacher wants to ensure that a student who guesses a response at random on any given question will achieve an expected score of zero for that question. What value of $b$ should the teacher choose?

   (b) Suppose there are 100 questions in all, and some student randomly guesses the answer to every one of them. What is his expected score, using the same value of $b$ as in part (a)?

   (c) A less clueless student is able to identify two false responses on every question, but then has to guess at random among the remaining three. What is her expected score on the test, again with the same value of $b$?

**5. (10 pts.) Chopping up DNA**

(a) In a certain biological experiment, a piece of DNA consisting of a linear sequence (or string) of 1000 nucleotides is subjected to bombardment by various enzymes. The effect of the bombardment is to randomly cut the string between pairs of adjacent nucleotides: each of the 999 possible cuts occurs independently and with probability $\frac{1}{250}$. What is the expected number of pieces into which the string is cut? Justify your calculation.

*Hint: Use linearity of expectation! If you do it this way, you can avoid a huge amount of messy calculation. Remember to justify the steps in your argument; i.e., do not appeal to "common sense"..*

(b) Suppose that the cuts are no longer independent, but highly correlated, so that when a cut occurs in a particular place other cuts close by are much more likely. The probability of each individual cut remains $\frac{1}{250}$. Does the expected number of pieces increase, decrease, or stay the same? Justify your answer with a precise explanation.

**6. (15 pts.) Martingales**

Consider a *fair game* in a casino: on each play, you may stake any amount $$S$; you win or lose with probability 1/2 each (all plays being independent); if you win you get your stake back plus $$S$; if you lose you lose your stake.

(a) What is the expected number of plays before your first win (including the play on which you win)?

(b) The following gambling strategy, known as the "martingale," was popular in European casinos in the 18th century: on the first play, stake $1; on the second play $2; on the third play $4; on the $k$th play $$2^{k-1}$. Stop (and leave the casino!) when you first win.

Show that, if you follow the martingale strategy, and assuming you have unlimited funds available, you will leave the casino $1 richer with probability 1. [Maybe this is why the strategy is banned in most modern casinos.]

(c) To discover the catch in this seemingly infallible strategy, let $X$ be the random variable that measures your maximum loss before winning (i.e., the amount of money you have lost *before* the play on which you win). Show that $\mathrm{E}(X) = \infty$. What does this imply about your ability to play the martingale strategy in practice?

(d) Colin and Diane enter the casino with $10 and $1,000,000 respectively. Both play the martingale strategy (leaving the casino either when they first win, or when they lack sufficient funds to place the next bet as required by the strategy). What is the probability that Colin wins? What is the probability that Diane wins?

**7. (15 pts.) A different birthday problem**

(a) In a company of $n$ people, what is the probability that exactly $k$ of them have a birthday on Christmas Day? [Your answer should contain a binomial coefficient, and should be given as a function of $n$ and $k$. Assume that birthdays are independently and uniformly distributed, and ignore the detail of leap years.]

(b) Now suppose $n = 500$. Compute the above probabilities (accurate to four decimal places) for $k = 0, 1, 2, 3, 4, 5, 6$. What is the expectation?

(c) The Poisson approximation for a binomial random variable with parameters $(n, p)$ (meaning that in $n$ independent trails, each results in a success with probability $p$ and failure with probability $1 - p$) is:

$$\Pr(X = i) \approx e^{-\lambda} \frac{\lambda^i}{i!}$$

where $X$ is the number of successes that occur after $n$ trials and $\lambda = np$.

Use the Poisson approximation to estimate the probabilities in part (b), again to four decimal places. How good is the approximation?

**8. (25 pts.) Book Problems**

(a) 6.1.20

(b) 6.1.54, 6.1.56 (a)

(c) 6.2.6

(d) 6.2.24

(e) 6.3.14

**9. (0 pts.) Optional Book Problems**

(a) 4.1.46

(b) 4.2.22

(c) 4.2.28

(d) 4.3.10

(e) 4.4.34

(f) 5.1.12

(g) 5.2.2

(h) 5.3.12