Today’s topics

• Graphs
  – Basics & types
  – Properties
  – Connectivity
  – Hamilton & Euler Paths

• Reading: Sections 8.1-8.5
Simple Graphs

- Correspond to symmetric, irreflexive binary relations $R$.
- A *simple graph* $G=(V,E)$ consists of:
  - a set $V$ of *vertices* or *nodes* ($V$ corresponds to the universe of the relation $R$),
  - a set $E$ of *edges* / *arcs* / *links*: unordered pairs of [distinct] elements $u,v \in V$, such that $uRv$.  

Visual Representation of a Simple Graph
Example of a Simple Graph

- Let $V$ be the set of states in the far-southeastern U.S.:
  - I.e., $V = \{\text{FL, GA, AL, MS, LA, SC, TN, NC}\}$
- Let $E = \{\{u,v\} | u \text{ adjoins } v\}$
  - $E = \{\{\text{FL,GA}\}, \{\text{FL,AL}\}, \{\text{FL,MS}\}, \{\text{FL,LA}\}, \{\text{GA,AL}\}, \{\text{AL,MS}\}, \{\text{MS,LA}\}, \{\text{GA,SC}\}, \{\text{GA,TN}\}, \{\text{SC,NC}\}, \{\text{NC,TN}\}, \{\text{MS,TN}\}, \{\text{MS,AL}\}\}$
Graph example

• Can the edge weights below be correct for any group of cities?
Multigraphs

• Like simple graphs, but there may be more than one edge connecting two given nodes.
• A multigraph $G=(V, E, f)$ consists of a set $V$ of vertices, a set $E$ of edges (as primitive objects), and a function $f: E \rightarrow \{\{u, v\} | u, v \in V \land u \neq v\}$.
• E.g., nodes are cities, edges are segments of major highways.
Pseudographs

- Like a multigraph, but edges connecting a node to itself are allowed. (*R may be reflexive.*)
- A *pseudograph* \( G=(V, E, f) \) where \( f:E \to \{\{u,v\} | u,v \in V\} \). Edge \( e \in E \) is a *loop* if \( f(e) = \{u,u\} = \{u\} \).
- *E.g.*, nodes are campsites in a state park, edges are hiking trails through the woods.
Directed Graphs

- Correspond to arbitrary binary relations $R$, which need not be symmetric.
- A directed graph $(V,E)$ consists of a set of vertices $V$ and a binary relation $E$ on $V$.
- E.g.: $V = \text{set of People}$, $E = \{(x,y) \mid x \text{ loves } y\}$
Directed Multigraphs

- Like directed graphs, but there may be more than one arc from a node to another.
- A directed multigraph $G=(V, E, f)$ consists of a set $V$ of vertices, a set $E$ of edges, and a function $f:E \rightarrow V \times V$.
- E.g., $V=$ web pages, $E=$ hyperlinks. The WWW is a directed multigraph...
Types of Graphs: Summary

- Summary of the book’s definitions.
- Keep in mind this terminology is not fully standardized across different authors...

<table>
<thead>
<tr>
<th>Term</th>
<th>Edge type</th>
<th>Multiple edges ok?</th>
<th>Self-loops ok?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simple graph</td>
<td>Undir.</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>Multigraph</td>
<td>Undir.</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>Pseudograph</td>
<td>Undir.</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Directed graph</td>
<td>Directed</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>Directed multigraph</td>
<td>Directed</td>
<td>Yes</td>
<td>Yes</td>
</tr>
</tbody>
</table>
§8.2: Graph Terminology

You need to learn the following terms:

- *Adjacent, connects, endpoints, degree, initial, terminal, in-degree, out-degree, complete, cycles, wheels, n-cubes, bipartite, subgraph, union.*
Adjacency

Let $G$ be an undirected graph with edge set $E$. Let $e \in E$ be (or map to) the pair $\{u,v\}$. Then we say:

• $u$, $v$ are adjacent / neighbors / connected.
• Edge $e$ is incident with vertices $u$ and $v$.
• Edge $e$ connects $u$ and $v$.
• Vertices $u$ and $v$ are endpoints of edge $e$. 
Degree of a Vertex

• Let $G$ be an undirected graph, $v \in V$ a vertex.
• The degree of $v$, $\deg(v)$, is its number of incident edges. (Except that any self-loops are counted twice.)
• A vertex with degree 0 is called isolated.
• A vertex of degree 1 is called pendant.
Handshaking Theorem

- Let $G$ be an undirected (simple, multi-, or pseudo-) graph with vertex set $V$ and edge set $E$. Then

$$\sum_{v \in V} \deg(v) = 2|E|$$

- **Corollary**: Any undirected graph has an even number of vertices of odd degree.
Directed Adjacency

• Let $G$ be a directed (possibly multi-) graph, and let $e$ be an edge of $G$ that is (or maps to) $(u,v)$. Then we say:
  – $u$ is adjacent to $v$, $v$ is adjacent from $u$
  – $e$ comes from $u$, $e$ goes to $v$.
  – $e$ connects $u$ to $v$, $e$ goes from $u$ to $v$
  – the initial vertex of $e$ is $u$
  – the terminal vertex of $e$ is $v$
Directed Degree

Let $G$ be a directed graph, $v$ a vertex of $G$.

- The in-degree of $v$, $\deg^-(v)$, is the number of edges going to $v$.
- The out-degree of $v$, $\deg^+(v)$, is the number of edges coming from $v$.
- The degree of $v$, $\deg(v) := \deg^-(v) + \deg^+(v)$, is the sum of $v$’s in-degree and out-degree.
Directed Handshaking Theorem

• Let $G$ be a directed (possibly multi-) graph with vertex set $V$ and edge set $E$. Then:

\[
\sum_{v \in V} \deg^{-}(v) = \sum_{v \in V} \deg^{+}(v) = \frac{1}{2} \sum_{v \in V} \deg(v) = |E|
\]

• Note that the degree of a node is unchanged by whether we consider its edges to be directed or undirected.
Special Graph Structures

Special cases of undirected graph structures:

- **Complete graphs** $K_n$
- **Cycles** $C_n$
- **Wheels** $W_n$
- **$n$-Cubes** $Q_n$
- **Bipartite graphs**
- **Complete bipartite graphs** $K_{m,n}$
Complete Graphs

- For any \( n \in \mathbb{N} \), a complete graph on \( n \) vertices, \( K_n \), is a simple graph with \( n \) nodes in which every node is adjacent to every other node: \( \forall u, v \in V: u \neq v \Leftrightarrow \{u, v\} \in E \).

Note that \( K_n \) has \( \sum_{i=1}^{n-1} i = \frac{n(n-1)}{2} \) edges.
Cycles

- For any \( n \geq 3 \), a cycle on \( n \) vertices, \( C_n \), is a simple graph where 
  \[ V = \{v_1, v_2, \ldots, v_n\} \]
  and 
  \[ E = \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}\} \].

How many edges are there in \( C_n \)?
Wheels

- For any \( n \geq 3 \), a wheel \( W_n \), is a simple graph obtained by taking the cycle \( C_n \) and adding one extra vertex \( v_{\text{hub}} \) and \( n \) extra edges \( \{ v_{\text{hub}}, v_1 \}, \{ v_{\text{hub}}, v_2 \}, \ldots, \{ v_{\text{hub}}, v_n \} \).

How many edges are there in \( W_n \)?
\textbf{\textit{n-cubes (hypercubes)}}

- For any \( n \in \mathbb{N} \), the hypercube \( Q_n \) is a simple graph consisting of two copies of \( Q_{n-1} \) connected together at corresponding nodes. \( Q_0 \) has 1 node.

\( Q_0 \) \hspace{1cm} \( Q_1 \) \hspace{1cm} \( Q_2 \) \hspace{1cm} \( Q_3 \) \hspace{1cm} \( Q_4 \)

\textit{Number of vertices:} \( 2^n \). \textit{Number of edges:} Exercise to try!
n-cubes (hypercubes)

- For any \( n \in \mathbb{N} \), the hypercube \( Q_n \) can be defined recursively as follows:
  - \( Q_0 = \{ \{ v_0 \}, \emptyset \} \) (one node and no edges)
  - For any \( n \in \mathbb{N} \), if \( Q_n = (V, E) \), where \( V = \{ v_1, \ldots, v_a \} \) and \( E = \{ e_1, \ldots, e_b \} \), then \( Q_{n+1} = (V \cup \{ v_1', \ldots, v_a' \}, E \cup \{ e_1', \ldots, e_b' \} \cup \{ \{ v_1, v_1' \}, \{ v_2, v_2' \}, \ldots, \{ v_a, v_a' \} \} ) \) where \( v_1', \ldots, v_a' \) are new vertices, and where if \( e_i = \{ v_j, v_k \} \) then \( e_i' = \{ v_j', v_k' \} \).
Bipartite Graphs

• **Def’n.:** A graph $G=(V,E)$ is *bipartite* (two-part) iff $V = V_1 \cap V_2$ where $V_1 \cup V_2 = \emptyset$ and $\forall e \in E: \exists v_1 \in V_1, v_2 \in V_2: e = \{v_1, v_2\}$.

• **In English:** The graph can be divided into two parts in such a way that all edges go between the two parts.

This definition can easily be adapted for the case of multigraphs and directed graphs as well.

Can represent with zero-one matrices.
Complete Bipartite Graphs

- For $m, n \in \mathbb{N}$, the complete bipartite graph $K_{m,n}$ is a bipartite graph where $|V_1| = m$, $|V_2| = n$, and $E = \{ \{v_1, v_2\} | v_1 \in V_1 \land v_2 \in V_2\}$.
  - That is, there are $m$ nodes in the left part, $n$ nodes in the right part, and every node in the left part is connected to every node in the right part.

$K_{4,3}$ has _____ nodes and _____ edges.
Subgraphs

- A subgraph of a graph $G=(V,E)$ is a graph $H=(W,F)$ where $W \subseteq V$ and $F \subseteq E$. 

![Diagram showing a subgraph of a larger graph](image)
Graph Unions

- The union $G_1 \cup G_2$ of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph $(V_1 \cup V_2, E_1 \cup E_2)$.
§8.3: Graph Representations & Isomorphism

- **Graph representations:**
  - Adjacency lists.
  - Adjacency matrices.
  - Incidence matrices.

- **Graph isomorphism:**
  - Two graphs are isomorphic iff they are identical except for their node names.
Adjacency Lists

- A table with 1 row per vertex, listing its adjacent vertices.

<table>
<thead>
<tr>
<th>Vertex</th>
<th>Adjacent Vertices</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>b, c</td>
</tr>
<tr>
<td>b</td>
<td>a, c, e, f</td>
</tr>
<tr>
<td>c</td>
<td>a, b, f</td>
</tr>
<tr>
<td>d</td>
<td></td>
</tr>
<tr>
<td>e</td>
<td>b</td>
</tr>
<tr>
<td>f</td>
<td>c, b</td>
</tr>
</tbody>
</table>
Directed Adjacency Lists

- 1 row per node, listing the terminal nodes of each edge incident from that node.
Adjacency Matrices

• A way to represent simple graphs
  – possibly with self-loops.
• Matrix $A = [a_{ij}]$, where $a_{ij}$ is 1 if $\{v_i, v_j\}$ is an edge of $G$, and is 0 otherwise.
• Can extend to pseudographs by letting each matrix elements be the number of links (possibly $>1$) between the nodes.
Graph Isomorphism

• Formal definition:
  – Simple graphs $G_1=(V_1, E_1)$ and $G_2=(V_2, E_2)$ are isomorphic iff $\exists$ a bijection $f: V_1 \rightarrow V_2$ such that $\forall a, b \in V_1$, $a$ and $b$ are adjacent in $G_1$ iff $f(a)$ and $f(b)$ are adjacent in $G_2$.
  – $f$ is the “renaming” function between the two node sets that makes the two graphs identical.
  – This definition can easily be extended to other types of graphs.
Graph Invariants under Isomorphism

*Necessary* but not *sufficient* conditions for $G_1=(V_1, E_1)$ to be isomorphic to $G_2=(V_2, E_2)$:

- We must have that $|V_1|=|V_2|$, and $|E_1|=|E_2|$.
- The number of vertices with degree $n$ is the same in both graphs.
- For every proper subgraph $g$ of one graph, there is a proper subgraph of the other graph that is isomorphic to $g$. 
Isomorphism Example

- If isomorphic, label the 2nd graph to show the isomorphism, else identify difference.
Are These Isomorphic?

- If isomorphic, label the 2nd graph to show the isomorphism, else identify difference.

- **Same # of vertices**
- **Same # of edges**
- **Different # of verts of degree 2!** (1 vs 3)
§8.4: Connectivity

- In an undirected graph, a *path of length n from u to v* is a sequence of adjacent edges going from vertex $u$ to vertex $v$.
- A path is a *circuit* if $u = v$.
- A path *traverses* the vertices along it.
- A path is *simple* if it contains no edge more than once.
Paths in Directed Graphs

- Same as in undirected graphs, but the path must go in the direction of the arrows.
An undirected graph is *connected* iff there is a path between every pair of distinct vertices in the graph.

**Theorem:** There is a *simple* path between any pair of vertices in a connected undirected graph.

**Connected component:** connected subgraph

A *cut vertex* or *cut edge* separates 1 connected component into 2 if removed.
Directed Connectedness

- A directed graph is *strongly connected* iff there is a directed path from $a$ to $b$ for any two verts $a$ and $b$.
- It is *weakly connected* iff the underlying *undirected* graph (i.e., with edge directions removed) is connected.
- Note *strongly* implies *weakly* but not vice-versa.
Paths & Isomorphism

• Note that connectedness, and the existence of a circuit or simple circuit of length $k$ are graph invariants with respect to isomorphism.
Counting Paths w Adjacency Matrices

- Let $A$ be the adjacency matrix of graph $G$.
- The number of paths of length $k$ from $v_i$ to $v_j$ is equal to $(A^k)_{i,j}$.
  - The notation $(M)_{i,j}$ denotes $m_{i,j}$ where $[m_{i,j}] = M$. 
§8.5: Euler & Hamilton Paths

- An **Euler circuit** in a graph $G$ is a simple circuit containing every edge of $G$.

- An **Euler path** in $G$ is a simple path containing every edge of $G$.

- A **Hamilton circuit** is a circuit that traverses each vertex in $G$ exactly once.

- A **Hamilton path** is a path that traverses each vertex in $G$ exactly once.
Bridges of Königsberg Problem

- Can we walk through town, crossing each bridge exactly once, and return to start?
Euler Path Theorems

- **Theorem:** A connected multigraph has an Euler circuit iff each vertex has even degree.
  - **Proof:**
    - ($\rightarrow$) The circuit contributes 2 to degree of each node.
    - ($\leftarrow$) By construction using algorithm on p. 580-581

- **Theorem:** A connected multigraph has an Euler path (but not an Euler circuit) iff it has exactly 2 vertices of odd degree.
  - One is the start, the other is the end.
Euler Circuit Algorithm

• Begin with any arbitrary node.
• Construct a simple path from it till you get back to start.
• Repeat for each remaining subgraph, splicing results back into original cycle.
Round-the-World Puzzle

• Can we traverse all the vertices of a dodecahedron, visiting each once?

Dodecahedron puzzle

Equivalent graph

Pegboard version
Hamiltonian Path Theorems

- **Dirac’s theorem**: If (but **not** only if) $G$ is connected, simple, has $n \geq 3$ vertices, and $\forall v \text{ deg}(v) \geq n/2$, then $G$ has a Hamilton circuit.

  - **Ore’s corollary**: If $G$ is connected, simple, has $n \geq 3$ nodes, and $\text{deg}(u) + \text{deg}(v) \geq n$ for every pair $u,v$ of non-adjacent nodes, then $G$ has a Hamilton circuit.