

Solutions - Homework 5

Problem 1

- a) Basically, the graph is transitive because \leq is transitive. Say there are three vertices x , y , and z . We know there are edges from x to y and from y to z . We want to show there is also an edge from x to z . Since $x \leq y$ and $y \leq z$, $x \leq z$. This being true, there is an edge from x to z .
- b) The graph is acyclic because we assumed that all values are distinct. Suppose we have three vertices, a , b , and c , with edges $(a, b), (b, c), (c, a)$ - that is, a cycle on three vertices. This means that $a \leq b \leq c \leq a$. Since we assumed all three values are distinct, however, this chain cannot exist. Therefore, there are no 3-cycles in the graph. We can combine this with part (a) to show there are no cycles in the graph.
- c) We have n vertices, each with a distinct weight. Considering them in sorted order, we can see that:
- Vertex 1 has $n - 1$ larger vertices.
 - Vertex 2 has $n - 2$ larger vertices

We can build a summation out of this. The correct number of edges in the graph is:

$$\sum_{i=1}^n (n - i) = \sum_{i=1}^n i = \frac{(n - 1)n}{2}$$

So the number of edges is $O(n^2)$.

- d) Topological sort takes all of the vertices and places them on a line with edges going from left to right. It can only be done, however, on a DAG (Directed Acyclic Graph). We proved in (b) that this graph is a DAG, and since there are edges wherever the weight of a vertex is less than another, Topological Sort will give us the sorted order. We use DFS to find the Topological Sort, so the algorithm is linear in the number of edges.

Problem 2

- a) Every MST is also an MMST.

We prove this using a proof by contradiction. First, assume that there is some MMST that has a better maximal edge than the MST. Recall, however, that Kruskal's algorithm builds a valid MST and adds edges in the order of their weights - meaning that the better edge would have been added before the max edge in the MST. Therefore, no MMST can have a better max edge than an MST and every MST is an MMST.

- b) No, every MMST is not an MST.

It's really easy to prove this by construction. Consider a graph with four vertices, three of which form a triangle. Give two of the edges of the triangle a weight of 1 and the other a

weight of 2. Now attach the fourth vertex to any of the other three with a much larger weight - maybe 10. There are two trivial spanning trees on this graph and both are MMSTs (the 10 edge must be included to get the fourth vertex), but only one is an MST.

Problem 3

Ideally, we want to convert an instance of this problem to something we know how to work with. Currently, however, the reliability of a path is based on the product of the reliabilities along it. There's a simple way to convert a product to a sum, however - just take the log of everything. If we take the log of all of the reliabilities, finding the best path is now a shortest path problem. We can just use Dijkstra's algorithm or the Bellman-Ford algorithm.