Ch. 7 - Pushdown Automata

A DFA = (Q, Σ, δ, q₀, F)

Modify DFA by adding a stack. New machine is called Pushdown Automata (PDA).

Definition: Nondeterministic PDA (NPDA) is defined by

\[ M = (Q, \Sigma, \Gamma, \delta, q₀, z, F) \]

where
Q is finite set of states
Σ is tape (input) alphabet
Γ is stack alphabet
q₀ is initial state
z - start stack symbol, (bottom of stack marker), z ∈ Γ
F ⊆ Q is set of final states.
δ: Q × (Σ ∪ {λ}) × Γ → finite subsets of Q × Γ*

Example of transitions

δ(q₁,a,b) = {(q₃,b),(q₄,ab),(q₆,λ)}

Meaning: If in state q₁ with “a” the current tape symbol and “b” the symbol on top of the stack, then pop “b”, and either

move to q₃ and push “b” on stack
move to q₄ and push “ab” on stack (“a” on top)
move to q₆

Transitions can be represented using a transition diagram.
The diagram for the above transitions is:

Each arc is labeled by a triple: x,y,z where x is the current input symbol, y is the top of stack symbol which is popped from the stack, and z is a string that is pushed onto the stack.

Instantaneous Description:

(q,w,u)

Notation to describe the current state of the machine (q), unread portion of the input string (w), and the current contents of the stack (u).
Description of a Move:

\[(q_1, aw, bx) \vdash (q_2, w, yx)\]

\[\text{iff}\]

**Definition** Let \(M=(Q, \Sigma, \Gamma, \delta, q_0, z, F)\) be a NPDA. \(L(M)=\{w \in \Sigma^* | (q_0, w, z) \vdash (p, \lambda, u), p \in F, u \in \Gamma^*\}\). The NPDA accepts all strings that start in \(q_0\) and end in a final state.

**Example:** \(L=\{a^n b^n | n \geq 0\}\), \(\Sigma = \{a, b\}\), \(\Gamma = \{z, a\}\)

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**Another Definition for Language Acceptance**

NPDA \(M\) accepts \(L(M)\) by empty stack:

\[L(M)=\{w \in \Sigma^* | (q_0, w, z) \vdash (p, \lambda, \lambda)\}\]
Example: \( L = \{ a^n b^m c^{n+m} | n, m > 0 \} \), \( \Sigma = \{ a, b, c \} \), \( \Gamma = \{ 0, z \} \)

Example: \( L = \{ w w^R | w \in \Sigma^+ \} \), \( \Sigma = \{ a, b \} \), \( \Gamma = \{ z, a, b \} \)

Example: \( L = \{ w w | w \in \Sigma^* \} \), \( \Sigma = \{ a, b \} \)

Examples for you to try on your own: (solutions are at the end of the handout).

- \( L = \{ a^n b^m | m > n, m, n > 0 \} \), \( \Sigma = \{ a, b \} \), \( \Gamma = \{ z, a \} \)
- \( L = \{ a^n b^{n+m} c^m | n, m > 0 \} \), \( \Sigma = \{ a, b, c \} \)
- \( L = \{ a^n b^{2n} | n > 0 \} \), \( \Sigma = \{ a, b \} \)
**Definition:** A PDA $M=(Q, \Sigma, \Gamma, \delta, q_0, z, F)$ is deterministic if for every $q \in Q$, $a \in \Sigma \cup \{\lambda\}$, $b \in \Gamma$

1. $\delta(q, a, b)$ contains at most 1 element
2. if $\delta(q, \lambda, b) \neq \emptyset$ then $\delta(q, c, b) = \emptyset$ for all $c \in \Sigma$

**Definition:** $L$ is DCFL iff $\exists$ DPDA $M$ s.t. $L = L(M)$.

**Examples:**

1. Previous pda for $\{a^n b^n | n \geq 0\}$ is deterministic?
2. Previous pda for $\{a^n b^m c^{n+m} | n, m > 0\}$ is deterministic?
3. Previous pda for $\{ww^R | w \in \Sigma^+, \Sigma = \{a, b\}\}$ is deterministic?
Example: $L = \{a^n b^m | m > n, m, n > 0\}$, $\Sigma = \{a, b\}$, $\Gamma = \{z, a\}$

Example: $L = \{a^n b^{n+m} c^m | n, m > 0\}$, $\Sigma = \{a, b, c\}$

Example: $L = \{a^n b^{2n} | n > 0\}$, $\Sigma = \{a, b\}$