Section: Pushdown Automata

Ch. 7 - Pushdown Automata

A DFA = \( (Q, \Sigma, \delta, q_0, F) \)
Modify DFA by adding a stack. New machine is called Pushdown Automata (PDA).

input tape

a a a a b b

tape head

head moves →

current state

0 1

stack

a
a
Z
Definition: Nondeterministic PDA (NPDA) is defined by

\[ M = (Q, \Sigma, \Gamma, \delta, q_0, z, F) \]

where

- \( Q \) is finite set of states
- \( \Sigma \) is tape (input) alphabet
- \( \Gamma \) is stack alphabet
- \( q_0 \) is initial state
- \( z \) - start stack symbol, \( z \in \Gamma \)
- \( F \subseteq Q \) is set of final states.

\( \delta : Q \times (\Sigma \cup \{\lambda\}) \times \Gamma \rightarrow \text{finite subsets of } Q \times \Gamma^* \)
Example of transitions

\[ \delta(q_1,a,b) = \{(q_3,b),(q_4,ab),(q_6,\lambda)\} \]

The diagram for the above transitions is:
Instantaneous Description:

\[(q,w,u)\]

Description of a Move:

\[(q_1, aw, bx) \vdash (q_2, w, yx)\]

iff

Definitions:

Let \( M = (Q, \Sigma, \Gamma, \delta, q_0, z, F) \) be a NPDA. \( L(M) = \{ w \in \Sigma^* \mid (q_0, w, z) \vdash^* (p, \lambda, u), p \in F, u \in \Gamma^* \} \). The NPDA accepts all strings that start in \( q_0 \) and end in a final state.
Example: $L = \{a^n b^n | n \geq 0 \}$, $\Sigma = \{a, b\}$, $\Gamma = \{z, a\}$
Another Definition for Language Acceptance

NPDA $M$ accepts $L(M)$ by empty stack:

$$L(M) = \{ w \in \Sigma^* | (q_0, w, z) \vdash^* (p, \lambda, \lambda) \}$$
Example: $L=\left\{ a^n b^m c^{n+m} \mid n, m > 0 \right\}$, 
$
\Sigma = \{a, b, c\}, \quad \Gamma = \{0, z\}$
Example: \( L = \{ w w^R | w \in \Sigma^+ \} \), \( \Sigma = \{ a, b \} \),
\( \Gamma = \{ z, a, b \} \)
Example: $L = \{ww|w \in \Sigma^*\}$, $\Sigma = \{a, b\}$

Examples for you to try on your own: (solutions are at the end of the handout).

- $L = \{a^n b^m|m > n, m, n > 0\}$, $\Sigma = \{a, b\}$, $\Gamma = \{z, a\}$
- $L = \{a^n b^{n+m} c^m|n, m > 0\}$, $\Sigma = \{a, b, c\}$
- $L = \{a^n b^{2n}|n > 0\}$, $\Sigma = \{a, b\}$
Definition: A PDA
\[ M = (Q, \Sigma, \Gamma, \delta, q_0, z, F) \] is deterministic if
for every \( q \in Q, \ a \in \Sigma \cup \{\lambda\}, \ b \in \Gamma \)

1. \( \delta(q, a, b) \) contains at most 1 element
2. if \( \delta(q, \lambda, b) \neq \emptyset \) then \( \delta(q, c, b) = \emptyset \) for all \( c \in \Sigma \)

Definition: \( L \) is DCFL iff \( \exists \) DPDA \( M \)
s.t. \( L = L(M) \).
Examples:

1. Previous pda for \( \{a^n b^n | n \geq 0\} \) is deterministic?

2. Previous pda for 
\( \{a^n b^m c^{n+m} | n, m > 0\} \) is deterministic?

3. Previous pda for 
\( \{ww^R | w \in \Sigma^+\}, \Sigma = \{a, b\} \) is deterministic?
Example: \( L = \{ a^n b^m | m > n, m, n > 0 \} \), \( \Sigma = \{ a, b \} \), \( \Gamma = \{ z, a \} \)

Example: \( L = \{ a^n b^{n+m} c^m | n, m > 0 \} \), \( \Sigma = \{ a, b, c \} \),

Example: \( L = \{ a^n b^{2n} | n > 0 \} \), \( \Sigma = \{ a, b \} \)