Section: Properties of Regular Languages

Example

\[ L = \{ a^n b a^n \mid n > 0 \} \]

Closure Properties

A set is closed over an operation if

\[ L_1, L_2 \in \text{class} \]
\[ L_1 \text{ op } L_2 = L_3 \]
\[ \Rightarrow L_3 \in \text{class} \]
\[ L_1 = \{ x \mid x \text{ is a positive even integer} \} \]

\( L \) is closed under

- addition?
- multiplication?
- subtraction?
- division?

Closure of Regular Languages

Theorem 4.1 If \( L_1 \) and \( L_2 \) are regular languages, then

\[ L_1 \cup L_2 \]
\[ L_1 \cap L_2 \]
\[ L_1 L_2 \]
\[ L_1^* \]
\[ \bar{L}_1 \]

are regular languages.
Proof(sketch)

$L_1$ and $L_2$ are regular languages

$\Rightarrow \exists$ reg. expr. $r_1$ and $r_2$ s.t.

$L_1 = L(r_1)$ and $L_2 = L(r_2)$

$r_1 + r_2$ is r.e. denoting $L_1 \cup L_2$

$\Rightarrow$ closed under union

$r_1 r_2$ is r.e. denoting $L_1 L_2$

$\Rightarrow$ closed under concatenation

$r_1^*$ is r.e. denoting $L_1^*$

$\Rightarrow$ closed under star-closure
complementation:
\[ L_1 \text{ is reg. lang.} \]
\[ \Rightarrow \exists \text{ DFA } M \text{ s.t. } L_1 = L(M) \]
Construct \( M' \) s.t.
- final states in \( M \) are nonfinal states in \( M' \)
- nonfinal states in \( M \) are final states in \( M' \)
\[ \Rightarrow \text{ closed under complementation} \]
intersection:

$L_1$ and $L_2$ are reg. lang.

$\Rightarrow \exists$ DFA $M_1$ and $M_2$ s.t.

$L_1 = L(M_1)$ and $L_2 = L(M_2)$

$M_1 = (Q, \Sigma, \delta_1, q_0, F_1)$

$M_2 = (P, \Sigma, \delta_2, p_0, F_2)$

Construct $M' = (Q', \Sigma, \delta', (q_0, p_0), F')$

$Q' = (Q \times P)$

$\delta'$:

$\delta'((q_i, p_j), a) = (q_k, p_l)$ if

$w \in L(M') \iff w \in L_1 \cap L_2$

$\Rightarrow$ closed under intersection
Example:
Regular languages are closed under

- reversal \( L^R \)
- difference \( L_1 - L_2 \)
- right quotient \( L_1 / L_2 \)
- homomorphism \( h(L) \)
Right quotient

Def: \( \frac{L_1}{L_2} = \{ x | xy \in L_1 \text{ for some } y \in L_2 \} \)

Example:

\[
L_1 = \{a^*b^* \cup b^*a^* \} \\
L_2 = \{b^n | n \text{ is even, } n > 0 \} \\
L_1/L_2 =
\]
Theorem If $L_1$ and $L_2$ are regular, then $L_1/L_2$ is regular.

Proof (sketch)

$\exists$ DFA $M=(Q,\Sigma,\delta,q_0,F)$ s.t. $L_1 = L(M)$.

Construct DFA $M'=(Q,\Sigma,\delta,q_0,F')$

For each state $i$ do

Make $i$ the start state (representing $L_i'$)

if $L_i' \cap L_2 \neq \emptyset$ then

put $q_i$ in $F'$ in $M'$

QED.
Homomorphism

Def. Let $\Sigma, \Gamma$ be alphabets. A homomorphism is a function

$$h: \Sigma \rightarrow \Gamma^*$$

Example:

$$\Sigma = \{a, b, c\}, \Gamma = \{0, 1\}$$

$$h(a) = 11$$
$$h(b) = 00$$
$$h(c) = 0$$

$$h(bc) =$$

$$h(ab^*) =$$
Questions about regular languages: 

L is a regular language.

• Given $L$, $\Sigma$, $w \in \Sigma^*$, is $w \in L$?

• Is $L$ empty?

• Is $L$ infinite?

• Does $L_1 = L_2$?
Identifying Nonregular Languages

If a language \( L \) is finite, is \( L \) regular?

If \( L \) is infinite, is \( L \) regular?

- \( L_1 = \{ a^n b^m | n > 0, m > 0 \} = aa^*bb^* \)
- \( L_2 = \{ a^n b^n | n > 0 \} \)
Prove that $L_2 = \{a^n b^n | n > 0\}$ is ?

• Proof:
Pumping Lemma: Let \( L \) be an infinite regular language. \( \exists \) a constant \( m > 0 \) such that any \( w \in L \) with \(|w| \geq m\) can be decomposed into three parts as \( w = xyz \) with

\[
\begin{align*}
|xy| &\leq m \\
|y| &\geq 1 \\
x y^i z &\in L \text{ for all } i \geq 0
\end{align*}
\]
To Use the Pumping Lemma to prove $L$ is not regular:

- **Proof by Contradiction.**
  
  Assume $L$ is regular.
  
  $\Rightarrow$ $L$ satisfies the pumping lemma.
  
  Choose a long string $w$ in $L$, $|w| \geq m$.
  
  Show that there is NO division of $w$ into $xyz$ (must consider all possible divisions) such that $|xy| \leq m$, $|y| \geq 1$ and $xy^iz \in L \ \forall \ i \geq 0$.
  
  The pumping lemma does not hold. Contradiction!
  
  $\Rightarrow$ $L$ is not regular. QED.
Example $L = \{a^n c b^n | n > 0\}$

$L$ is not regular.

- **Proof:**
  Assume $L$ is regular.
  $\Rightarrow$ the pumping lemma holds.
  Choose $w =$
Example $L = \{a^n b^{n+s} c^s | n, s > 0 \}$

$L$ is not regular.

- **Proof:**
  
  Assume $L$ is regular.
  
  $\Rightarrow$ the pumping lemma holds.

  Choose $w = \quad$

  So the partition is:
Example $\Sigma = \{a, b\}$,
$L = \{w \in \Sigma^* \mid n_a(w) > n_b(w)\}$

$L$ is not regular.

- Proof:
  Assume $L$ is regular.
  $\Rightarrow$ the pumping lemma holds.
  Choose $w =$
  So the partition is:
Example $L = \{a^3 b^n c^{n-3} | n > 3\}$

$L$ is not regular.
To Use Closure Properties to prove $L$ is not regular:

- **Proof Outline:**
  
  Assume $L$ is regular.
  
  Apply closure properties to $L$ and other regular languages, constructing $L'$ that you know is not regular.
  
  closure properties $\Rightarrow L'$ is regular.
  
  Contradiction!
  
  $L$ is not regular. QED.
Example $L=\{a^3b^n c^{n-3} | n > 3\}$

$L$ is not regular.

- Proof: (proof by contradiction)
  Assume $L$ is regular.
  Define a homomorphism $h : \Sigma \rightarrow \Sigma^*$
  
  \[
  h(a) = a \quad h(b) = a \quad h(c) = b
  \]

  $h(L) =$
Example \( L = \{a^n b^m a^m | m \geq 0, n \geq 0\} \)

\( L \) is not regular.

• Proof: (proof by contradiction)
  Assume \( L \) is regular.
Example:  $L_1 = \{a^n b^n a^n | n > 0\}$

$L_1$ is not regular.