Today's topics

- Propositional equivalences
- Predicate logic

- Reading: Sections 1.2-1.3

Propositional Equivalence (§1.2)

Two *syntactically* (*i.e.,* textually) different compound propositions may be the *semantically* identical (*i.e.,* have the same meaning). We call them *equivalent*. Learn:

- Various *equivalence rules* or *laws*.
- How to *prove* equivalences using *symbolic derivations*.

Tautologies and Contradictions

A *tautology* is a compound proposition that is **true** *no matter what* the truth values of its atomic propositions are!

Ex. $p \vee \neg p$ [What is its truth table?]

A *contradiction* is a compound proposition that is **false** no matter what! *Ex.* $p \wedge \neg p$ [Truth table?]

Other compound props. are *contingencies*.

Logical Equivalence

Compound proposition p is *logically equivalent* to compound proposition q, written $p \Leftrightarrow q$, **IFF** the compound proposition $p \leftrightarrow q$ is a tautology.

Compound propositions p and q are logically equivalent to each other **IFF** p and q contain the same truth values as each other in <u>all</u> rows of their truth tables.

Proving Equivalence via Truth Tables

Ex. Prove that $p \vee q \Leftrightarrow \neg(\neg p \wedge \neg q)$.

Equivalence Laws

- These are similar to the arithmetic identities you may have learned in algebra, but for propositional equivalences instead.
- They provide a pattern or template that can be used to match all or part of a much more complicated proposition and to find an equivalence for it.

Equivalence Laws - Examples

- *Identity*: $p \wedge \mathbf{T} \Leftrightarrow p$ $\quad p \vee \mathbf{F} \Leftrightarrow p$
- *Domination*: $p \vee \mathbf{T} \Leftrightarrow \mathbf{T}$ $\quad p \wedge \mathbf{F} \Leftrightarrow \mathbf{F}$
- *Idempotent*: $p \vee p \Leftrightarrow p$ $\quad p \wedge p \Leftrightarrow p$
- *Double negation:* $\neg\neg p \Leftrightarrow p$
- *Commutative:* $p \vee q \Leftrightarrow q \vee p$ $\quad p \wedge q \Leftrightarrow q \wedge p$
- *Associative:* $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$
  $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$

More Equivalence Laws

- *Distributive*: $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$
  $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$
- *De Morgan's*:
  $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$
  $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$
- *Trivial tautology/contradiction*:
  $p \vee \neg p \Leftrightarrow \mathbf{T}$ $\quad p \wedge \neg p \Leftrightarrow \mathbf{F}$

Augustus De Morgan (1806-1871)

Defining Operators via Equivalences

Using equivalences, we can *define* operators in terms of other operators.

- Exclusive or: $p \oplus q \Leftrightarrow (p \vee q) \wedge \neg(p \wedge q)$
  $p \oplus q \Leftrightarrow (p \wedge \neg q) \vee (q \wedge \neg p)$
- Implies: $p \rightarrow q \Leftrightarrow \neg p \vee q$
- Biconditional: $p \leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$
  $p \leftrightarrow q \Leftrightarrow \neg(p \oplus q)$

Review: Propositional Logic (§§1.1-1.2)

- Atomic propositions: p, q, r, …
- Boolean operators: $\neg \wedge \vee \oplus \rightarrow \leftrightarrow$
- Compound propositions: $s :\equiv (p \wedge \neg q) \vee r$
- Equivalences: $p \wedge \neg q \Leftrightarrow \neg(p \rightarrow q)$
- Proving equivalences using:
  - Truth tables.
  - Symbolic derivations. $p \Leftrightarrow q \Leftrightarrow r$ …

Predicate Logic (§1.3)

- *Predicate logic* is an extension of propositional logic that permits concisely reasoning about whole *classes* of entities.
- Propositional logic (recall) treats simple *propositions* (sentences) as atomic entities.
- In contrast, *predicate* logic distinguishes the *subject* of a sentence from its *predicate*.
  - Remember these English grammar terms?

Practical Applications of Predicate Logic

- It is the basis for clearly expressed formal specifications for any complex system.
- It is basis for *automatic theorem provers* and many other Artificial Intelligence systems.
  - *E.g.* automatic program verification systems.
- Predicate-logic like statements are supported by some of the more sophisticated *database query engines* and *container class libraries*
  - these are types of programming tools.

Subjects and Predicates

- In the sentence "The dog is sleeping":
  - The phrase "the dog" denotes the *subject* - the *object* or *entity* that the sentence is about.
  - The phrase "is sleeping" denotes the *predicate*- a property that is true **of** the subject.
- In predicate logic, a *predicate* is modeled as a *function* $P(\cdot)$ from objects to propositions.
  - $P(x)$ = "x is sleeping" (where x is any object).

More About Predicates

- Convention: Lowercase variables x, y, z... denote objects/entities; uppercase variables P, Q, R… denote propositional functions (predicates).
- Keep in mind that the *result of applying* a predicate P to an object x is the *proposition* $P(x)$. But the predicate P **itself** (*e.g.* P="is sleeping") is **not** a proposition (not a complete sentence).
  - *E.g.* if $P(x)$ = "x is a prime number", $P(3)$ is the *proposition* "3 is a prime number."

Propositional Functions

- Predicate logic *generalizes* the grammatical notion of a predicate to also include propositional functions of **any** number of arguments, each of which may take **any** grammatical role that a noun can take.
  - *E.g.* let $P(x,y,z)$ = "x gave y the grade z", then if x="Mike", y="Mary", z="A", then $P(x,y,z)$ = "Mike gave Mary the grade A."

Universes of Discourse (U.D.s)

- The power of distinguishing objects from predicates is that it lets you state things about *many* objects at once.
- E.g., let $P(x)$="$x+1>x$". We can then say, "For *any* number x, $P(x)$ is true" instead of $(\mathbf{0}+1>\mathbf{0}) \wedge (\mathbf{1}+1>\mathbf{1}) \wedge (\mathbf{2}+1>\mathbf{2}) \wedge \ldots$
- The collection of values that a variable x can take is called x's *universe of discourse*.

Quantifier Expressions

- *Quantifiers* provide a notation that allows us to *quantify* (count) *how many* objects in the univ. of disc. satisfy a given predicate.
- "∀" is the FOR∀LL or *universal* quantifier. $\forall x\, P(x)$ means *for all* x in the u.d., *P* holds.
- "∃" is the ∃XISTS or *existential* quantifier. $\exists x\, P(x)$ means there *exists* an x in the u.d. (that is, 1 or more) such that $P(x)$ is true.

The Universal Quantifier ∀

- Example:
  Let the u.d. of x be parking spaces at Duke.
  Let $P(x)$ be the *predicate* "x is full."
  Then the *universal quantification of* $P(x)$, $\forall x\, P(x)$, is the *proposition:*
  - "All parking spaces at Duke are full."
  - *i.e.*, "Every parking space at Duke is full."
  - *i.e.*, "For each parking space at Duke, that space is full."

The Existential Quantifier ∃

- Example:
  Let the u.d. of x be parking spaces at Duke.
  Let $P(x)$ be the *predicate* "x is full."
  Then the *existential quantification of* $P(x)$, $\exists x\, P(x)$, is the *proposition*:
  - "Some parking space at Duke is full."
  - "There is a parking space at Duke that is full."
  - "At least one parking space at Duke is full."

Free and Bound Variables

- An expression like $P(x)$ is said to have a *free variable* x (meaning, x is undefined).
- A quantifier (either ∀ or ∃) *operates* on an expression having one or more free variables, and *binds* one or more of those variables, to produce an expression having one or more *bound variables*.

Example of Binding

- $P(x,y)$ has 2 free variables, x and y.
- $\forall x\ P(x,y)$ has 1 free variable, and one bound variable. [Which is which?]
- "$P(x)$, where x=3" is another way to bind x.
- An expression with <u>zero</u> free variables is a bona-fide (actual) proposition.
- An expression with <u>one or more</u> free variables is still only a predicate: *e.g.* let $Q(y) = \forall x\ P(x,y)$

Nesting of Quantifiers

Example: Let the u.d. of x & y be people.

Let $L(x,y)$="x likes y" (a predicate w. 2 f.v.'s)

Then $\exists y\ L(x,y)$ = "There is someone whom x likes." (A predicate w. 1 free variable, x)

Then $\forall x\ (\exists y\ L(x,y))$ =

"Everyone has someone whom they like."

(A __________ with ___ free variables.)

Quantifier Exercise

If $R(x,y)$="x relies upon y," express the following in unambiguous English:

$\forall x(\exists y\ R(x,y))$=

$\exists y(\forall x\ R(x,y))$=

$\exists x(\forall y\ R(x,y))$=

$\forall y(\exists x\ R(x,y))$=

$\forall x(\forall y\ R(x,y))$=