Today's topics

- DNF
- Predicate logic: Nested Quantifiers

- Reading: Minesweeper notes, Section 1.4

Natural language is ambiguous!

- "Everybody likes somebody."
  - For everybody, there is somebody they like,
    - $\forall x\, \exists y\, Likes(x,y)$
  - or, there is somebody (a popular person) whom everyone likes?
    - $\exists y\, \forall x\, Likes(x,y)$
- "Somebody likes everybody."
  - Same problem: Depends on context, emphasis.

Game Theoretic Semantics

- Thinking in terms of a competitive game can help you tell whether a proposition with nested quantifiers is true.
- The game has two players, <u>both with the same knowledge</u>:
  - Verifier: Wants to demonstrate that the proposition is true.
  - Falsifier: Wants to demonstrate that the proposition is false.
- The Rules of the Game "Verify or Falsify":
  - Read the quantifiers from <u>left to right</u>, picking values of variables.
  - When you see "∀", the falsifier gets to select the value.
  - When you see "∃", the verifier gets to select the value.
- If the verifier <u>can always win</u>, then the proposition is true.
- If the falsifier <u>can always win</u>, then it is false.

Let's Play, "Verify or Falsify!"

Let $B(x,y)$:≡ "x's birthday is followed within 7 days by y's birthday."

Suppose I claim that among you:
$\underline{\forall x}\, \exists y\, B(x,y)$

Your turn, as falsifier:
You pick any $x \rightarrow$ *(so-and-so)*

$\underline{\exists y}\, B(\text{so-and-so},y)$

My turn, as verifier:
I pick any $y \rightarrow$ *(such-and-such)*

$B(\text{so-and-so},\text{such-and-such})$

- Let's play it in class.
- Who wins this game?
- What if I switched the quantifiers, and I claimed that $\exists y\, \forall x\, B(x,y)$? Who wins in that case?

Still More Conventions

- Sometimes the universe of discourse is restricted within the quantification, *e.g.*,
  - $\forall x{>}0\ P(x)$ is shorthand for "For all x that are greater than zero, $P(x)$." $=\forall x\ (x{>}0 \rightarrow P(x))$
  - $\exists x{>}0\ P(x)$ is shorthand for "There is an x greater than zero such that $P(x)$." $=\exists x\ (x{>}0 \wedge P(x))$

More to Know About Binding

- $\forall x\ \exists x\ P(x)$ - x is not a free variable in $\exists x\ P(x)$, therefore the $\forall x$ binding isn't used.
- $(\forall x\ P(x)) \wedge \mathrm{Q}(x)$ - The variable x is outside of the *scope* of the $\forall x$ quantifier, and is therefore free. Not a complete proposition!
- $(\forall x\ P(x)) \wedge (\exists x\ \mathrm{Q}(x))$ – This is legal, because there are 2 different x's!

Quantifier Equivalence Laws

- Definitions of quantifiers: If u.d.=a,b,c,…
  $\forall x\ P(x) \Leftrightarrow P(\mathrm{a}) \wedge P(\mathrm{b}) \wedge P(\mathrm{c}) \wedge \ldots$
  $\exists x\ P(x) \Leftrightarrow P(\mathrm{a}) \vee P(\mathrm{b}) \vee P(\mathrm{c}) \vee \ldots$
- From those, we can prove the laws:
  $\forall x\ P(x) \Leftrightarrow \neg\exists x\ \neg P(x)$
  $\exists x\ P(x) \Leftrightarrow \neg\forall x\ \neg P(x)$
- Which *propositional* equivalence laws can be used to prove this?

More Equivalence Laws

- $\forall x\ \forall y\ P(x,y) \Leftrightarrow \forall y\ \forall x\ P(x,y)$
  $\exists x\ \exists y\ P(x,y) \Leftrightarrow \exists y\ \exists x\ P(x,y)$
- $\forall x\ (P(x) \wedge Q(x)) \Leftrightarrow (\forall x\ P(x)) \wedge (\forall x\ Q(x))$
  $\exists x\ (P(x) \vee Q(x)) \Leftrightarrow (\exists x\ P(x)) \vee (\exists x\ Q(x))$
- Exercise:
  See if you can prove these yourself.
  - What propositional equivalences did you use?

More Notational Conventions

- Quantifiers bind as loosely as needed: parenthesize $\forall x\ \ P(x) \wedge Q(x)$
- Consecutive quantifiers of the same type can be combined: $\forall x\ \forall y\ \forall z\ P(x,y,z) \Leftrightarrow \forall x,y,z\ P(x,y,z)$ or even $\forall xyz\ P(x,y,z)$
- All quantified expressions can be reduced to the canonical *alternating* form $\forall x_1 \exists x_2 \forall x_3 \exists x_4 \ldots\ P(x_1, x_2, x_3, x_4, \ldots)$

Defining New Quantifiers

As per their name, quantifiers can be used to express that a predicate is true of any given *quantity* (number) of objects.

Define $\exists! x\ P(x)$ to mean "$P(x)$ is true of *exactly one* x in the universe of discourse."

$\exists! x\ P(x) \Leftrightarrow \exists x\ (P(x) \wedge \neg\exists y\ (P(y) \wedge y \neq x))$

"There is an x such that $P(x)$, where there is no y such that $P(y)$ and y is other than x."

Some Number Theory Examples

- Let u.d. = the *natural numbers* 0, 1, 2, …
- "A number x is *even*, $E(x)$, if and only if it is equal to 2 times some other number."
- "A number is *prime*, $P(x)$, iff it's greater than 1 and it isn't the product of any two non-unity numbers."

Goldbach's Conjecture (unproven)

Using $E(x)$ and $P(x)$ from previous slide,

$\forall E(x{>}2): \exists P(p),P(q): p+q = x$

or, with more explicit notation:

$\forall x\ [x{>}2 \wedge E(x)] \rightarrow$
$\exists p\ \exists q\ P(p) \wedge P(q) \wedge p+q = x.$

"Every even number greater than 2 is the sum of two primes."

Calculus Example

- One way of precisely defining the calculus concept of a *limit*, using quantifiers:

$$\left(\lim_{x \to a} f(x) = L\right) \Leftrightarrow$$

Deduction Example

- Definitions:
  - s :≡ Socrates (ancient Greek philosopher);
  - $H(x)$:≡ "x is human";
  - $M(x)$:≡ "x is mortal".
- Premises:
  - $H(\mathrm{s})$ — *Socrates is human.*
  - $\forall x\, H(x) \rightarrow M(x)$ — *All humans are mortal.*

Deduction Example Continued

Some valid conclusions you can draw:

$H(\mathrm{s}) \rightarrow M(\mathrm{s})$ **[Instantiate universal.]** *If Socrates is human then he is mortal.*

$\neg H(\mathrm{s}) \vee M(\mathrm{s})$ *Socrates is inhuman or mortal.*

$H(\mathrm{s}) \wedge (\neg H(\mathrm{s}) \vee M(\mathrm{s}))$ *Socrates is human, and also either inhuman or mortal.*

$(H(\mathrm{s}) \wedge \neg H(\mathrm{s})) \vee (H(\mathrm{s}) \wedge M(\mathrm{s}))$ **[Apply distributive law.]**

$\mathbf{F} \vee (H(\mathrm{s}) \wedge M(\mathrm{s}))$ **[Trivial contradiction.]**

$H(\mathrm{s}) \wedge M(\mathrm{s})$ **[Use identity law.]**

$M(\mathrm{s})$ *Socrates is mortal.*

Another Example

- Definitions: $H(x)$:≡ "x is human"; $M(x)$:≡ "x is mortal"; $G(x)$:≡ "x is a god"
- Premises:
  - $\forall x\, H(x) \rightarrow M(x)$ ("Humans are mortal") and
  - $\forall x\, G(x) \rightarrow \neg M(x)$ ("Gods are immortal").
- Show that $\neg\exists x\, (H(x) \wedge G(x))$ ("No human is a god.")

The Derivation

- $\forall x\ H(x) \rightarrow M(x)$ and $\forall x\ G(x) \rightarrow \neg M(x)$.
- $\forall x\ \neg M(x) \rightarrow \neg H(x)$ **[Contrapositive.]**
- $\forall x\ [G(x) \rightarrow \neg M(x)] \wedge [\neg M(x) \rightarrow \neg H(x)]$
- $\forall x\ G(x) \rightarrow \neg H(x)$ **[Transitivity of →.]**
- $\forall x\ \neg G(x) \vee \neg H(x)$ **[Definition of →.]**
- $\forall x\ \neg(G(x) \wedge H(x))$ **[DeMorgan's law.]**
- $\neg \exists x\ G(x) \wedge H(x)$ **[An equivalence law.]**

End of §1.3-1.4, Predicate Logic

- From these sections you should have learned:
  - Predicate logic notation & conventions
  - Conversions: predicate logic ↔ clear English
  - Meaning of quantifiers, equivalences
  - Simple reasoning with quantifiers
- Upcoming topics:
  - Introduction to proof-writing.
  - Then: Set theory –
    - a language for talking about collections of objects.