Today's topics

- Counting
  - Sum rule
  - Product rule
  - Tree diagrams
  - Inclusion/exclusion
- Reading: Sections 4.1
- Upcoming
  - Permutations & Combinations

Combinatorics

- The study of the number of ways to put things together into various combinations.
- *E.g.* In a contest entered by 100 people,
  - how many different top-10 outcomes could occur?
- *E.g.* If a password is 6-8 letters and/or digits,
  - how many passwords can there be?

Sum and Product Rules (§4.1)

- Let m be the number of ways to do task 1 and n the number of ways to do task 2,
  - with each number independent of how the other task is done,
  - and also assume that no way to do task 1 simultaneously also accomplishes task 2.
- Then, we have the following rules:
  - The *sum rule*: The task "do either task 1 or task 2, but not both" can be done in $m+n$ ways.
  - The *product rule*: The task "do both task 1 and task 2" can be done in mn ways.

Set Theoretic Version

- If A is the set of ways to do task 1, and B the set of ways to do task 2, and if A and B are disjoint, then:
  - The ways to do either task 1 or 2 are $A \cup B$, and $|A \cup B| = |A| + |B|$
  - The ways to do both task 1 and 2 can be represented as $A \times B$, and $|A \times B| = |A| \cdot |B|$

IP Address Example

- Some facts about the Internet Protocol, version 4:
  - Valid computer addresses are in one of 3 types:
    - A *class A* IP address contains a 7-bit "netid" ≠ 1^7, and a 24-bit "hostid"
    - A *class B* address has a 14-bit netid and a 16-bit hostid.
    - A *class C* addr. Has 21-bit netid and an 8-bit hostid.
  - The 3 classes have distinct headers (0, 10, 110)
  - Hostids that are all 0s or all 1s are not allowed.
- How many valid computer addresses are there?

e.g., duke.edu is 152.3.233.10

Inclusion-Exclusion Principle (§§4.1 & 6.5)

- Suppose that $k \leq m$ of the ways of doing task 1 also simultaneously accomplish task 2.
  - And thus are also ways of doing task 2.
- Then, the number of ways to accomplish "Do either task 1 or task 2" is $m+n-k$.
- Set theory: If A and B are not disjoint, then $|A \cup B| = |A| + |B| - |A \cap B|$.
  - If they are disjoint, this simplifies to $|A|+|B|$.

Inclusion/Exclusion Example

- Some hypothetical rules for passwords:
  - Passwords must be 2 characters long.
  - Each character must be a letter a-z, a digit 0-9, or one of the 10 punctuation characters !@#$%^&*().
  - Each password must contain at least 1 digit or punctuation character.

Setup of Problem

- A legal password has a digit or puctuation character in position 1 **or** position 2.
  - These cases overlap, so the principle applies.
- (# of passwords w. OK symbol in position #1) = (10+10)·(10+10+26)
- (# w. OK sym. in pos. #2): also 20·46
- (# w. OK sym both places): 20·20
- Answer:

Pigeonhole Principle (§4.2)

- A.k.a. the "Dirichlet drawer principle"
- If $\geq k+1$ objects are assigned to k places, then at least 1 place must be assigned ≥ 2 objects.
- In terms of the assignment function:
  - If $f{:}A \rightarrow B$ and $|A| \geq |B|+1$, then some element of B has ≥ 2 preimages under f.
    - I.e., f is not one-to-one.

Example of Pigeonhole Principle

- There are 101 possible numeric grades (0%-100%) rounded to the nearest integer.
  - Also, there are >101 students in this class.
- Therefore, there must be at least one (rounded) grade that will be shared by at least 2 students at the end of the semester.
  - I.e., the function from students to rounded grades is *not* a one-to-one function.

Fun Pigeonhole Proof (Ex. 4, p.314)

- **Theorem:** $\forall n \in \mathbf{N}$, $\exists$ a multiple $m>0$ of n ∋ m has only 0's and 1's in its decimal expansion!
- **Proof:** Consider the $n+1$ decimal integers 1, 11, 111, …, $\underbrace{1 \cdots 1}_{n+1}$. They have only n possible residues mod n. So, take the difference of two that have the same residue. The result is the answer! □

Another Fun Example

- Suppose that next June, the Durham Bulls play at least 1 game a day, but ≤45 games total. Show there must be some sequence of consecutive days in June during which they play *exactly* 14 games.
  - **Proof:** Let a_j be the number of games played on or before day j. Then, $a_1,\ldots,a_{30} \in \mathbf{Z}^+$ is a sequence of 30 distinct integers with $1 \leq a_j \leq 45$. Therefore $a_1+14,\ldots,a_{30}+14$ is a sequence of 30 distinct integers with $15 \leq a_j+14 \leq 59$. Thus, $(a_1,\ldots,a_{30},a_1+14,\ldots,a_{30}+14)$ is a sequence of 60 integers from the set {1,..,59}. By the Pigeonhole Principle, two of them must be equal, but $a_i \neq a_j$ for $i \neq j$. So, $\exists ij$: $a_i = a_j+14$. Thus, 14 games were played on days $a_j+1, \ldots, a_i$.