Today's topics

- Counting
  - Generalized Pigeonhole Principle
  - Permutations
  - Combinations
  - Binomial Coefficients
- Reading: Sections 4.2-4.3, 4.4
- Upcoming
  - Probability

Generalized Pigeonhole Principle

- If N objects are assigned to k places, then at least one place must be assigned at least $\lceil N/k \rceil$ objects.
- *E.g.*, there are N=280 students in this class. There are k=52 weeks in the year.
  - Therefore, there must be at least 1 week during which at least $\lceil 280/52 \rceil = \lceil 5.38 \rceil = 6$ students in the class have a birthday.

Proof of G.P.P.

- By contradiction. Suppose every place has $< \lceil N/k \rceil$ objects, thus $\leq \lceil N/k \rceil - 1$.
- Then the total number of objects is at most

$$k\left(\left\lceil \frac{N}{k} \right\rceil - 1\right) < k\left(\left(\frac{N}{k} + 1\right) - 1\right) = k\left(\frac{N}{k}\right) = N$$

- So, there are less than N objects, which contradicts our assumption of N objects! □

G.P.P. Example

- Given: There are 280 students in the class.
  - Without knowing anybody's birthday, what is the largest value of n for which we can prove using the G.P.P. that at least n students must have been born in the same month?
- Answer:

Permutations (§4.3)

- A *permutation* of a set S of objects is a sequence that contains each object in S exactly once.
  - An ordered arrangement of r distinct elements of S is called an *r-permutation of S*.
- The number of r-permutations of a set with $n=|S|$ elements is

$$P(n,r) = n(n-1)\ldots(n-r+1) = n!/(n-r)!$$

Permutation Example

- You are in a silly action movie where there is a bomb, and it is your job to disable it by cutting wires to the trigger device. There are 10 wires to the device. If you cut exactly the right three wires, in exactly the right order, you will disable the bomb, otherwise it will explode! If the wires all look the same, what are your chances of survival?

Combinations (§4.3)

- An r-combination of elements of a set S is simply a subset $T\subseteq S$ with r members, $|T|=r$.
- The number of r-combinations of a set with $n=|S|$ elements is

$$C(n,r) = \binom{n}{r} = \frac{P(n,r)}{P(r,r)} = \frac{n!/(n-r)!}{r!} = \frac{n!}{r!(n-r)!}$$

- Note that $C(n,r) = C(n, n-r)$
  - Because choosing the r members of T is the same thing as choosing the $n-r$ non-members of T.

Combination Example

- How many distinct 7-card hands can be drawn from a standard 52-card deck?
  - The order of cards in a hand doesn't matter.

Binomial coefficients

- Binomial coefficient $\binom{n}{r}$
  - Given $(1 + x)^r$
  - What are the coefficients of x^r ?

$$(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r$$
$$= \binom{n}{0} x^0 + \binom{n}{1} x^1 + \cdots + \binom{n}{n} x^n$$

  - Show for $(1 + x)^4$

Important Theorems on Binomials

- Binomial thereom
  - In the expansion of $(1 + x)^n$, the coefficient of x^r equals $C(n,r)$

$$\binom{n}{r} = C(n,r) = \frac{n^{\underline{r}}}{r!}$$

- Pascal's Identity

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

- Proofs?