Today's topics

- Recurrence relations
  - Stating recurrences
  - LiHoReCoCo
- Reading: Sections 6.1-6.2
- Upcoming
  - Graphs

§6.1: Recurrence Relations

- A *recurrence relation* (R.R., or just *recurrence*) for a sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more previous elements $a_0, \ldots, a_{n-1}$ of the sequence, for all $n \geq n_0$.
  - *I.e.*, just a recursive definition, without the base cases.
- A particular sequence (described non-recursively) is said to *solve* the given recurrence relation if it is consistent with the definition of the recurrence.
  - A given recurrence relation may have many solutions.

Recurrence Relation Example

- Consider the recurrence relation

  $a_n = 2a_{n-1} - a_{n-2}$ $(n \geq 2)$.
- Which of the following are solutions?

  $a_n = 3n$

  $a_n = 2^n$

  $a_n = 5$

Example Applications

- Recurrence relation for growth of a bank account with P% interest per given period:

  $M_n = M_{n-1} + (P/100)M_{n-1}$
- Growth of a population in which each organism yields 1 new one every period starting 2 time periods after its birth.

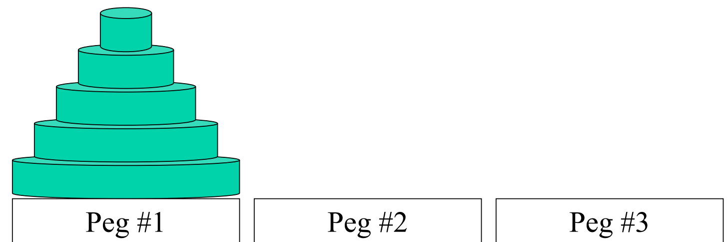
  $P_n = P_{n-1} + P_{n-2}$ (Fibonacci relation)

Solving Compound Interest RR

- $M_n = M_{n-1} + (P/100)M_{n-1}$
  $= (1 + P/100)\ M_{n-1}$
  $= r\ M_{n-1}$ (let $r = 1 + P/100$)
  $= r\ (r\ M_{n-2})$
  $= r \cdot r \cdot (r\ M_{n-3})$ …and so on to…
  $= r^n\ M_0$

Tower of Hanoi Example

- Problem: Get all disks from peg 1 to peg 2.
  - Rules: (a) Only move 1 disk at a time.
  - (b) Never set a larger disk on a smaller one.



Hanoi Recurrence Relation

- Let H_n = # moves for a stack of n disks.
- Here is the optimal strategy:
  - Move top $n-1$ disks to spare peg. (H_{n-1} moves)
  - Move bottom disk. (1 move)
  - Move top $n-1$ to bottom disk. (H_{n-1} moves)
- **Note that:** $H_n = 2H_{n-1} + 1$
  - The # of moves is described by a Rec. Rel.

Solving Tower of Hanoi RR

$H_n = 2\ H_{n-1} + 1$
$= 2\ (2\ H_{n-2} + 1) + 1 \quad = 2^2\ H_{n-2} + 2 + 1$
$= 2^2(2\ H_{n-3} + 1) + 2 + 1 \quad = 2^3\ H_{n-3} + 2^2 + 2 + 1$
…
$= 2^{n-1}\ H_1 + 2^{n-2} + \ldots + 2 + 1$
$= 2^{n-1} + 2^{n-2} + \ldots + 2 + 1$ (since $H_1 = 1$)
$= \sum_{i=0}^{n-1} 2^i$
$= 2^n - 1$

Another R.R. Example

- Find a R.R. & initial conditions for the number of bit strings of length n without two consecutive 0s.
- We can solve this by breaking down the strings to be counted into cases that end in 0 and in 1.
  - For each ending in 0, the previous bit must be 1, and before that comes any qualifying string of length $n-2$.
  - For each string ending in 1, it starts with a qualifying string of length $n-1$.
- Thus, $a_n = a_{n-1} + a_{n-2}$. (Fibonacci recurrence.)
  - The initial conditions are: $a_0 = 1$ (ε), $a_1 = 2$ (0 and 1).

($n-2$ bits)	1	0

($n-1$ bits)	1

Yet another R.R. example…

- Give a recurrence (and base cases) for the number of n-digit decimal strings containing an *even* number of 0 digits.
- Can break down into the following cases:
  - Any valid string of length $n-1$ digits, with any digit 1-9 appended.
  - Any *invalid* string of length $n-1$ digits, + a 0.
- $a_n = 9a_{n-1} + (10^{n-1} - a_{n-1}) = 8a_{n-1} + 10^{n-1}$.
  - Base cases: $a_0 = 1$ (ε), $a_1 = 9$ (1-9).

§6.2: Solving Recurrences

General Solution Schemas

- A *linear homogeneous recurrence of degree k with constant coefficients* ("k-LiHoReCoCo") is a rec. rel. of the form

  $$a_n = c_1 a_{n-1} + \ldots + c_k a_{n-k},$$

  where the c_i are all real, and $c_k \neq 0$.
- The solution is uniquely determined if k initial conditions $a_0 \ldots a_{k-1}$ are provided.

Solving LiHoReCoCos

- Basic idea: Look for solutions of the form $a_n = r^n$, where r is a constant.
- This requires solving the *characteristic equation*:

  $r^n = c_1 r^{n-1} + \ldots + c_k r^{n-k}$, *i.e.*,

  $r^k - c_1 r^{k-1} - \ldots - c_k = 0$ (rearrange & × by r^{k-n})
- The solutions r to this equation are called the *characteristic roots* of the LiHoReCoCo.
  - They can yield an explicit formula for the sequence.

Solving 2-LiHoReCoCos

- Consider an arbitrary 2-LiHoReCoCo:
  $a_n = c_1 a_{n-1} + c_2 a_{n-2}$
- It has the characteristic equation (C.E.):
  $r^2 - c_1 r - c_2 = 0$
- **Theorem 1:** If the CE has 2 roots $r_1 \neq r_2$, then the solutions to the RR are given by:
  $a_n = \alpha_1 r_1^{\,n} + \alpha_2 r_2^{\,n}$ for $n \geq 0$
  for any and all constants α_1, α_2.

Special case: $a_n = r_1^{\,n}$ and $a_n = r_2^{\,n}$ are, of course, solutions.

Example

- Solve the recurrence $a_n = a_{n-1} + 2a_{n-2}$ given the initial conditions $a_0 = 2$, $a_1 = 7$.
- Solution: Use theorem 1:
  - We have $c_1 = 1$, $c_2 = 2$
  - The characteristic equation is: $r^2 - r - 2 = 0$
  - Solve it: $r = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-2)}}{2(1)} = \frac{1 \pm \sqrt{9}}{2} = \frac{1 \pm 3}{2} = 2 \text{ or } -1.$
  - so, $r = 2$ or $r = -1$.
  - So, $a_n = \alpha_1 2^n + \alpha_2 (-1)^n$.

(Using the quadratic formula here.)

$ax^2 + bx + c = 0 \Leftrightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Example Continued…

- To find α_1 and α_2, just solve the equations for the initial conditions a_0 and a_1:
  $a_0 = 2 = \alpha_1 2^0 + \alpha_2 (-1)^0$
  $a_1 = 7 = \alpha_1 2^1 + \alpha_2 (-1)^1$
  Simplifying, we have the pair of equations:
  $2 = \alpha_1 + \alpha_2$
  $7 = 2\alpha_1 - \alpha_2$
  which we can solve easily by substitution:
  $\alpha_2 = 2 - \alpha_1$; $7 = 2\alpha_1 - (2 - \alpha_1) = 3\alpha_1 - 2$;
  $9 = 3\alpha_1$; $\alpha_1 = 3$; $\alpha_2 = -1$.
- Using α_1 and α_2, our final answer is: $a_n = 3 \cdot 2^n - (-1)^n$

Check: $\{a_{n \geq 0}\} = 2, 7, 11, 25, 47, 97 \ldots$

Proof of Theorem 1

- Proof that $a_n = \alpha_1 r_1^{\,n} + \alpha_2 r_2^{\,n}$ is always a solution:
  - We know $r_1^2 = c_1 r_1 + c_2$ and $r_2^2 = c_1 r_2 + c_2$.
  - Now we can show the proposed sequence satisfies the recurrence $a_n = c_1 a_{n-1} + c_2 a_{n-2}$:

$$c_1 a_{n-1} + c_2 a_{n-2} = c_1(\alpha_1 r_1^{\,n-1} + \alpha_2 r_2^{\,n-1}) + c_2(\alpha_1 r_1^{\,n-2} + \alpha_2 r_2^{\,n-2})$$
$$= \alpha_1 r_1^{\,n-2}(c_1 r_1 + c_2) + \alpha_2 r_2^{\,n-2}(c_1 r_2 + c_2)$$
$$= \alpha_1 r_1^{\,n-2} r_1^{\,2} + \alpha_2 r_2^{\,n-2} r_2^{\,2} = \alpha_1 r_1^{\,n} + \alpha_2 r_2^{\,n} = a_n. \quad \square$$

- Can complete the proof by showing that for any initial conditions, we can find corresponding α's.
  - But it turns out this goes through only if $r_1 \neq r_2$.

The Case of Degenerate Roots

- Now, what if the C.E. $r^2 - c_1 r - c_2 = 0$ has only 1 root r_0?
- **Theorem 2:** Then,

  $a_n = \alpha_1 r_0{}^n + \alpha_2 n r_0{}^n$, for all $n \geq 0$,

  for some constants α_1, α_2.

Degenerate Root Example

- Solve $a_n = 6a_{n-1} - 9a_{n-2}$ with $a_0=1$, $a_1=6$.
- The C.E. is: $r^2 - 6r + 9 = 0$.
  - Note that $b^2 - 4ac = (-6)^2 - 4 \cdot 1 \cdot 9 = 36 - 36 = 0$.
    - Therefore, there is only one root, namely $-b/2a = -(-6)/2 = 3$.

k-LiHoReCoCos

- Consider a k-LiHoReCoCo: $$a_n = \sum_{i=1}^{k} c_i a_{n-i}$$
- It's C.E. is: $$r^k - \sum_{i=1}^{k} c_i r^{k-i} = 0$$
- **Theorem 3:** If this has k distinct roots r_i, then the solutions to the recurrence are of the form:

  $$a_n = \sum_{i=1}^{k} \alpha_i r_i^n$$

  for all $n \geq 0$, where the α_i are constants.

Degenerate *k*-LiHoReCoCos

- Suppose there are t roots $r_1, \ldots, r_t$ with multiplicities $m_1, \ldots, m_t$. Then:

  $$a_n = \sum_{i=1}^{t} \left(\sum_{j=0}^{m_i - 1} \alpha_{i,j} n^j \right) r_i^n$$

  for all $n \geq 0$, where all the α are constants.

LiNoReCoCos

- Linear *nonhomogeneous* RRs with constant coefficients may (unlike Li**Ho**ReCoCos) contain some terms $F(n)$ that depend *only* on n (and *not* on any a_i's). General form:

$$\underbrace{a_n = c_1 a_{n-1} + \ldots + c_k a_{n-k}}\, + F(n)$$

The *associated homogeneous recurrence relation* (associated LiHoReCoCo).

Solutions of LiNoReCoCos

- A useful theorem about LiNoReCoCos:
  - If $a_n = p(n)$ is any *particular* solution to the LiNoReCoCo: $a_n = \left(\sum_{i=1}^{k} c_i a_{n-i}\right) + F(n)$
  - Then *all* of its solutions are of the form: $a_n = p(n) + h(n)$, where $a_n = h(n)$ is *any* solution to the associated homogeneous RR $a_n = \left(\sum_{i=1}^{k} c_i a_{n-i}\right)$

LiNoReCoCo Example

- Find all solutions to $a_n = 3a_{n-1}+2n$. Which solution has $a_1 = 3$?
  - Notice this is a 1-LiNoReCoCo. Its associated 1-LiHoReCoCo is $a_n = 3a_{n-1}$, whose solutions are all of the form $a_n = \alpha 3^n$. Thus the solutions to the original problem are all of the form $a_n = p(n) + \alpha 3^n$. So, all we need to do is find one $p(n)$ that works.

Trial Solutions

- If the extra terms $F(n)$ are a degree-t polynomial in n, you should try a general degree-t polynomial as the particular solution $p(n)$.
- This case: $F(n)$ is linear so try $a_n = cn + d$.

  $cn+d = 3(c(n-1)+d) + 2n$ (for all n)

  $(2c+2)n + (2d-3c) = 0$ (collect terms)

  So $c = -1$ and $d = -3/2$.

  So $a_n = -n - 3/2$ is a solution.
- **Check:** $a_{n\geq 1} = \{-5/2, -7/2, -9/2, \ldots\}$

Finding a Desired Solution

- From the previous, we know that all general solutions to our example are of the form:

  $a_n = -n - 3/2 + \alpha 3^n$.

  Solve this for α for the given case, $a_1 = 3$:

  $3 = -1 - 3/2 + \alpha 3^1$

  $\alpha = 11/6$
- The answer is $a_n = -n - 3/2 + (11/6)3^n$.

Double Check Your Answer!

- Check the base case, a_1=3:

  $a_n = -n - 3/2 + (11/6)3^n$

  $a_1 = -1 - 3/2 + (11/6)3^1$

  $= -2/2 - 3/2 + 11/2 = -5/2 + 11/2 = 6/2 = 3$
- Check the recurrence, $a_n = 3a_{n-1} + 2n$:

  $-n - 3/2 + (11/6)3^n = 3[-(n-1) - 3/2 + (11/6)3^{n-1}] + 2n$

  $= 3[-n - 1/2 + (11/6)3^{n-1}] + 2n$

  $= -3n - 3/2 + (11/6)3^n + 2n = -n - 3/2 + (11/6)3^n$ ■