Bayes Nets

CPS 170
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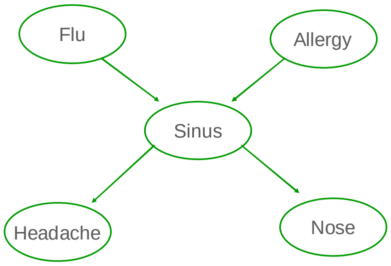
Conditional Independence

- Suppose we know the following:
  - The flu causes sinus inflammation
  - Allergies cause sinus inflammation
  - Sinus inflammation causes a runny nose
  - Sinus inflammation causes headaches
- How are these connected?

Causal Structure



Knowing sinus separates the variables from each other.

Conditional Independence

- We say that two variables, A and B, are conditionally independent given C if:
  - P(A|BC) = P(A|C)
- How does this help?
- We store only a conditional probability table (CPT) of each variable given its parents
- Naïve Bayes (e.g. Spam Assassin) is a special case of this!

Notation Reminder

- P(A|B) is a conditional prob. distribution
  - It is a function!
  - P(A=true|B=true), P(A=true|B=false), P(A=false|B=True), P(A=false|B=true)
- P(A|b) is a probability distribution, function
- P(a|B) is a function, not a distribution
- P(a|b) is a number

Getting More Formal

- What is a Bayes net?
  - A directed acyclic graph (DAG)
  - Parents chosen such that

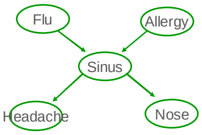
$$P(x_1 ... x_n) = \prod_i P(x_i \mid \text{parents}(x_i))$$

  - Given the parents, each variable is independent of non-descendents
  - For each node X_i, store $P(X_i|\text{parents}(X_i))$
  - Represent as table called a CPT

Real Applications of Bayes Nets

- Diagnosis of lymph node disease
- Used in Microsoft office and Windows
  - http://www.research.microsoft.com/research/dtg/
- Used by robots to identify meteorites to study
- Study the human genome:Alex Hartemink et al.
- Many other applications…

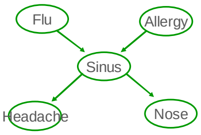
Space Efficiency



- Entire joint as 32 (31) entries
  - P(H|S),P(N|S) have 4 (2)
  - P(S|AF) has 8 (4)
  - P(A) has 2 (1)
  - Total is 20 (10)
- This can require exponentially less space
- Space problem is solved for “most” problems

Atomic Event Probabilities

$$P(x_1 \ldots x_n) = \prod_i P(x_i \mid \text{parents}(x_i))$$



Note that this is guaranteed true if we construct net incrementally, so that for each new variable added, we connect all influencing Variables as parents

Doing Things the Hard Way

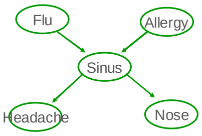
$$P(f \mid h) = \frac{P(fh)}{P(h)} = \frac{\sum_{SAN} P(fhSAN)}{\sum_{SANF} P(hSANF)}$$

defn. of conditional probability

marginalization

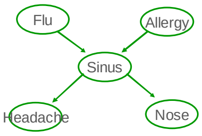
Doing this naïvely, we need to sum over all atomic events defined over these variables. There are exponentially many of these.

Working Smarter I



$$\begin{aligned}
P(hSANF) &= P(hN \mid SAF)P(SAF) \\
&= P(hN \mid S)P(SAF) \\
&= P(h \mid S)P(N \mid S)P(SAF) \\
&= P(h \mid S)P(N \mid S)P(S \mid AF)P(AF) \\
&= P(h \mid S)P(N \mid S)P(S \mid AF)P(A)P(F)
\end{aligned}$$

Working Smarter II



$$\begin{aligned}
P(h) &= \sum_{SANF} P(hSANF) \\
&= \sum_{SANF} P(hN \mid SAF)P(SAF) \\
&= \sum_{NS} P(hN \mid S) \sum_{AF} P(SAF) \\
&= \sum_{S} P(h \mid S) \sum_{N} P(N \mid S) \sum_{AF} P(SAF) \\
&= \sum_{S} P(h \mid S) \sum_{N} P(N \mid S) \sum_{AF} P(S \mid AF)P(A)P(F)
\end{aligned}$$

Potential for exponential reduction in computation.

Checkpoint

- BNs can give us an exponential reduction in the space required to represent a joint distribution.
- Storage is exponential in largest parent set.
- Claim: Parent sets are often reasonable.
- Claim: Inference cost is often reasonable.
- Question: Can we quantify relationship between structure and inference cost?

Computational Efficiency

$$\sum_{SANF} P(hSANF) = \sum_{SANF} P(h \mid S)P(N \mid S)P(S \mid AF)P(A)P(F)$$
$$= \sum_{S} P(h \mid S)\sum_{N} P(N \mid S)\sum_{AF} P(S \mid AF)P(A)P(F)$$

The distributive law allows us to decompose the sum.

Potential for an exponential reduction in computation costs.

What Is a Bayes Net, Really?

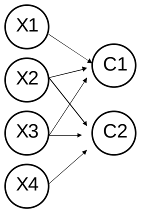
- A Bayes net is a data structure (with associated algorithms) for fast manipulation of probability distributions
- Bayes nets solve computational problems
- Real problems are solved by is *not* a method
  - Modeling distribution of features is a method
  - Exploiting independence is a method
- Bayes nets represent; they do not solve
- Q: How often can a bnet solve a computational efficiency problem?

Now the Bad News…

- In full generality: Inference is NP-hard
- Decision problem: Is $P(X_i) > 0.5$?
- We reduce from 3SAT
- 3SAT variables map to BN variables
- Clauses become variables with the corresponding SAT variables as parents

Reduction

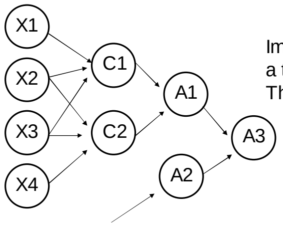
$$(\overline{X}_1 \vee X_2 \vee X_3) \wedge (\overline{X}_2 \vee X_3 \vee X_4) \wedge \ldots$$



Problem: What if we have a large number of clauses? How does this fit into our decision problem framework?

And Trees

We could make a single variable which is the AND of all of our clauses, but this would have CPT that is exponential in the number of clauses.



Implement as a tree of ANDs. This is polynomial.

Is BN Inference NP Complete?

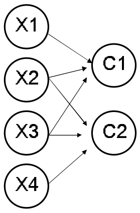
- Can show that BN inference is #P hard
- #P is counting the number of satisfying assignments
- Idea: Assign variables uniform probability
- Probability of conjunction of clauses tells us how many assignments are satisfying

Checkpoint

- BNs can be very compact
- Worst case: Inference is intractable
- Hope that worst is case:
  - Avoidable
  - Easily characterized in some way

(Undirected) Trees

- Are the structures from our reduction trees?

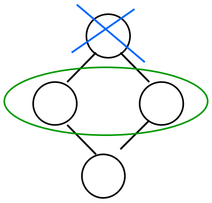


No!

Dealing With General DAGs

- Q: How does graphical structure relate to our ability to push in summations over variables?
- A:
  - We relate summations to graph operations
  - Summing out a variable =
    - Removing node(s) from DAG
    - Creating new replacement node
  - Relate graph properties to computational effiency

Variable Elimination

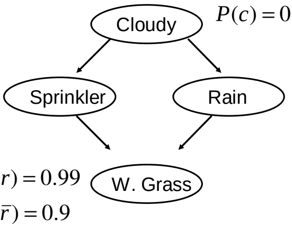


Recall that in variable elimination for CSPs, we eliminated variables and created new supervariables

BN Variable Elimination

- The same trick applies to Bayes nets
- Observation:
  - Every variable elimination ordering corresponds to a rearrangement of the summation in the marginalization computation

Another Example Network



$P(c)=0.5$

$P(s\mid c)=0.1$

$P(s\mid \bar{c})=0.5$

$P(r\mid c)=0.8$

$P(r\mid \bar{c})=0.2$

$P(w\mid sr)=0.99$

$P(w\mid s\bar{r})=0.9$

$P(w\mid \bar{s}r)=0.9$

$P(w\mid \bar{s}\bar{r})=0.0$

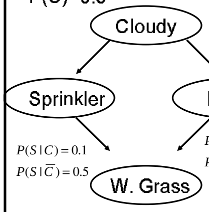
Marginal Probabilities

Suppose we want P(W):

$$P(W)=\sum_{CSR}P(CSRW)$$
$$=\sum_{CSR}P(C)P(S\mid C)P(R\mid C)P(W\mid RS)$$
$$=\sum_{SR}P(W\mid RS)\sum_{C}P(S\mid C)P(C)P(R\mid C)$$

Eliminating Cloudy

P(C)=0.5



$P(S\mid C)=0.1$

$P(S\mid \overline{C})=0.5$

$P(R\mid C)=0.8$

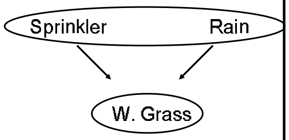
$P(R\mid \overline{C})=0.2$

$P(sr)=0.5*0.1*0.8+0.5*0.5*0.2=0.09$

$P(s\bar{r})=0.5*0.1*0.2+0.5*0.5*0.8=0.21$

$P(\bar{s}r)=0.5*0.9*0.8+0.5*0.5*0.2=0.41$

$P(\bar{s}\bar{r})=0.5*0.9*0.2+0.5*0.5*0.8=0.29$



$$P(W)=\sum_{CSR}P(CSRW)$$
$$=\sum_{CSR}P(C)P(S\mid C)P(R\mid C)P(W\mid RS)$$
$$=\sum_{SR}P(W\mid RS)\sum_{C}P(S\mid C)P(C)P(R\mid C)$$

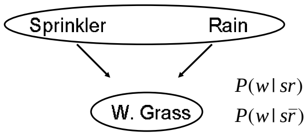
Eliminating Sprinkler/Rain

$P(sr)=0.09$

$P(s\bar{r})=0.21$

$P(\bar{s}r)=0.41$

$P(\bar{s}\bar{r})=0.29$



$P(w\mid sr)=0.99$

$P(w\mid s\bar{r})=0.9$

$P(w\mid \bar{s}r)=0.9$

$P(w\mid \bar{s}\bar{r})=0.0$

$$P(w)=\sum_{SR}P(w\mid RS)P(RS)$$
$$=0.09*0.99+0.21*0.9+0.41*0.9+0.29*0$$
$$=0.6471$$

Dealing With Evidence

Suppose we have observed that the grass is wet?
What is the probability that it has rained?

$$P(R\mid W)=\alpha P(RW)$$
$$=\alpha\sum_{CS}P(CSRW)$$
$$=\alpha\sum_{CS}P(C)P(S\mid C)P(R\mid C)P(W\mid RS)$$
$$=\alpha\sum_{C}P(R\mid C)P(C)\sum_{S}P(S\mid C)P(W\mid RS)$$

Is there a more clever way to deal with w?

The Variable Elimination Algorithm

```
Elim(bn, query)
If bn.vars = query
   return bn
Else
   x = pick_variable(bn)
   newbn.vars = bn.vars - x
   newbn.vars = newbn.vars - neighbors(x)
   newbn.vars = newbn.vars + newvar
   newbn.vars(newvar).function =
              $\sum_{X}\prod_{Y\in X\cup neighbors(X)}$bn.vars(Y) .function
   return(elim(newbn, query))
```

Efficiency of Variable Elimination

- Exponential in the largest domain size of new variables created
- Equivalently: Exponential in largest function created by pushing in summations

- Linear for trees
- Almost linear for almost trees ☺
- (See examples on board…)

Beyond Variable Elimination

- Variable elimination must be rerun for every new query
- Possible to compile a Bayes net into a new data structure to make repeated queries more efficient
  - Note that inference in trees is linear
  - Define a cluster tree where
    - Clusters = sets of original variables
    - Can infer original probs from cluster probs
- For networks w/o good elimination schemes
  - Sampling
  - Variational methods

Facts About Variable Elimination

- Picking variables in optimal order is NP hard
- For some networks, there will be no elimination ordering that results in a poly time solution (Must be the case unless P=NP)
- Polynomial for trees
- Need to get a little fancier if there are a large number of query variables or evidence variables

Bayes Net Summary

- Bayes net = data structure for joint distribution

- Can give exponential reduction in storage

- Variable elimination:
  - simple, elegant method
  - efficient for many networks

- For some networks, must use approximation