NP Hardness & CSPs
CPS 170

Ron Parr

NP-hardness

- Many problems in AI are NP-hard (or worse)
- What does this mean?
- These are some of the hardest problems in CS
- Identifying a problem as NP hard means:
  - – You probably shouldn't waste time trying to find a polynomial time solution
  - – If you find a polynomial time solution, either
    - You have a bug
    - Find a place on your shelf for your Turing award
- NP hardness is a major triumph (and failure) for computer science theory

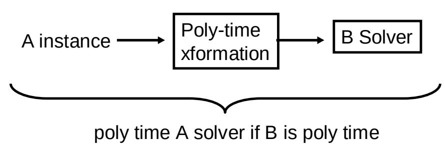
What is the class NP?

- A class of decision problems (Yes/No)
- Solutions can be verified in polynomial time
- Examples:
  - – Graph coloring:

  - – Sortedness: [1 2 3 4 5 8 7]

What is NP completeness?

- All NP *complete* problems can be "reduced" to each other in polynomial time
- What is a reduction?
  - – Use one problem to solve another
  - – A is reduced to B, if we can use B to solve A:



Why care about NP-completeness?

- Solving any one NP-complete problem gives you the key to all others
- All NP-complete problems are, in a sense, *equivalent*
- Insight into solving any one gives you insight into solving a vast array of problems of extraordinary practical and economic significance

Proving NP Completeness

- Want to prove problem C is NP complete
  - – Show that C is in NP
  - – Find known NP complete problem reducible to C
  - – Is graph color NP-complete?
    - Prove that graph coloring is in NP
      - – Verify solution in poly time
      - – Easy
    - Reduce known NP complete problem to TSPs
      - – Much more challenging
      - – Reduction to SAT

The First NP Complete Problem (Cook 1971)

- SAT:

$$(X_1 \vee \overline{X}_7 \vee X_{13}) \wedge (\overline{X}_2 \vee X_{12} \vee X_{25}) \wedge \ldots$$

- Want to find an assignment to all variables that makes this expression evaluate to true
- NP-complete for clauses of size 3 or greater
- How would you prove this?

What is NP Hardness?

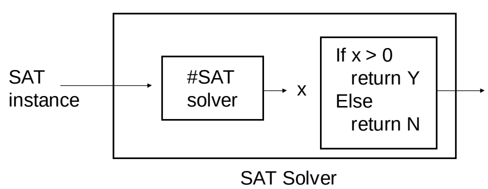
- NP hardness is weaker than NP completeness
- NP hard if an NP complete problem is reducible to it
- NP completeness = NP hardness + NP membership
- Consider the problem #SAT
  - How many satisfying assignments to:

$$(X_1 \vee \overline{X}_7 \vee X_{13}) \wedge (\overline{X}_2 \vee X_{12} \vee X_{25}) \wedge \ldots$$

  - Is this in NP?
  - Is it NP-hard?

#SAT is NP-hard

- Theorem: #SAT is NP hard
- Proof:
  - Reduce SAT to #SAT



NP-Completeness Summary

- NP-completeness tells us that a problem belongs to class of similar, hard problems.
- What if you find that a problem is NP hard?
  - Look for good approximations
  - Find different measures of complexity
  - Look for tractable subclasses
  - Use heuristics

CSPs

- What is a CSP?
- One view: Search with special goal criteria
- CSP definition (general):
  - Variables $X_1,\ldots,X_n$
  - Variable X_i has domain D_i
  - Constaints $C_1,\ldots,C_m$
  - Solution: Each variable gets a value from its domain such that no constraints violated
- CSP examples…
  - http://4c.ucc.ie/~tw/csplib/

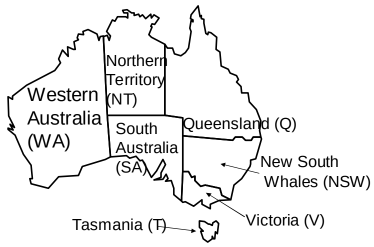
Our Restricted View

- Variables $X_1,\ldots,X_n$
- A binary constraint, lists permitted assignments to pairs of variables
- A binary constraint between binary variables is a table of size 4, listing legal assignments for all 4 combinations.
- A k-ary constraint lists legal assignments to k variables at a time.
- How large is a k-ary constraint for binary variables?

Note: More expressive languages are often used.

CSP Example

Graph coloring:

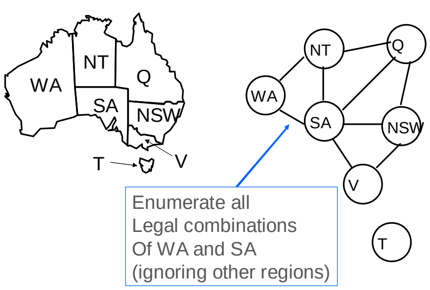


Problem: Assign Red, Green and Blue so that no 2 adjacent regions have the same color.

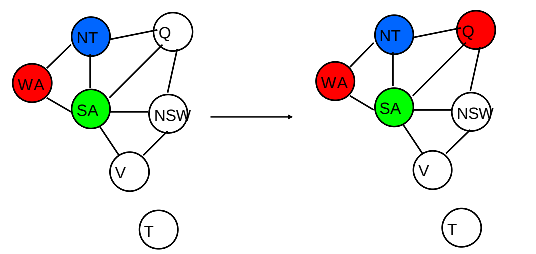
Example Contd.

- Variables: {WA, NT, Q, SA, NSW, V, T}
- Domains: {R,G,B}
- Constraints:
  For WA – NT:{(R,G), (R,B), (G,B), (G,R), (B,R), (B,G)}
- We have a table for each adjacent pair
- Are our constraints binary?
- Can every CSP be viewed as a graph problem?

Constraint Graph



CSPs as Search



Nodes: Partial Assignments Actions: Make Assignments

Backtracking

- Backtracking is the most obvious (and widely used) method for solving CSPs:
  – Search forward by assigning values to variables
  – If stuck, undo the most recent assignment and try again
  – Repeat until satisfying assignment found or all combinations tried
- Embellishments
  – Methods for picking next variable to assign
    - Most constrained
    - Least constrained
  – Backjumping

NP-Completeness of CSPs

- Are CSPs in NP?
- Are they NP-hard?

- CSPs and graph coloring are equivalent
  - Convert any graph coloring problem to CSP
  - Convert any CSP to graph coloring
- Graph coloring is NP-complete
- CSPs are NP-complete
- End of the story or just the beginning?

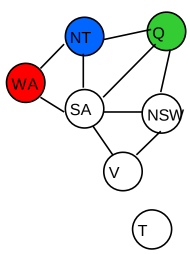
Issues

- What are good heuristics?
  - Often good to think of this as a local search
  - Focus on choosing actions carefully, instead of pruning nodes carefully
- Can we develop heuristics that apply to the entire class of problems, not just specific instances?
- What's the best we can hope for?

Constraint Graphs

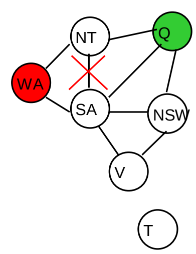
- Constraint graphs are important because they capture the structural relationships between the variables
- IMPORTANT CONCEPT:
  Not all instances of a hard problem class are hard
  - Structural features give insight into hardness
  - Group problems within class by structural features
  - New measure of problem complexity

Node Consistency



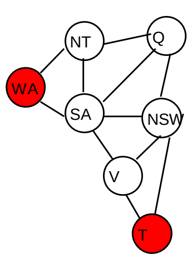
- Check all nodes for inconsistencies
- For each node, there must exist at least one valid assignment given assignments to neighbors
- Rules out some bad assignments quickly

Arc Consistency



- Check all arcs for inconsistencies
- For each value at the start, there must exist a consistent value at the terminus
- Catches many inconsistencies
- Can use to iteratively reduce number of possible assignments to each variable
  (constraint propagation)

Generalized Arc Consistency



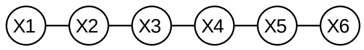
- k-consistency
  - Consider sets of k variables
  - For each setting of a k-1 subset
  - Must exist a consistent setting for the k^{th} variable
- Check for more distant influences
- 1-consistency = node consistency
- 2 consistency = arc consistency

Is this 3-consistent?

Facts About Arc Consistency

- What if a graph with n variables is n-consistent?
- What is the worst-case cost of checking n-consistency?

Linear Constraint Structures



Are these easy or hard?

Suppose our chain is arc consistent…

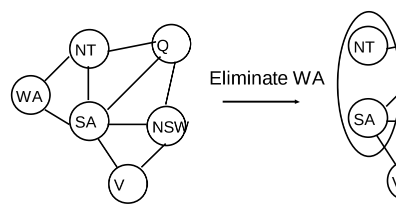
Properties of Chains

Theorem: Arc consistent linear constraint graphs are n consistent.

Properties of Trees

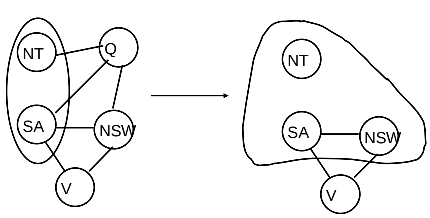
Theorem: Arc consistent constraint trees are n consistent.

Variable Elimination



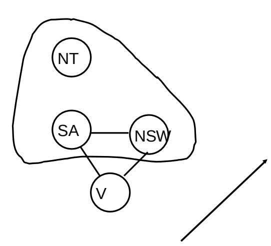
Domain(NT,SA) = {(blue, green), (blue, red), (green, blue), (green, red), (red, blue), (red, green)}

Eliminate Q



Domain(NT,SA,NSW) = {(blue, green, blue), (blue, red, blue), (red, blue, red), (red, green, red), (green, blue, green), (green, red, green)}

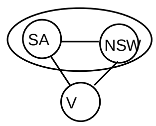
Simplify



Domain(SA, NSW) = {(blue, green), (blue, red), (green, blue), (green, red), (red, blue), (red, green)}

Domain(NT,SA,NSW) = {(blue, green, blue), (blue, red, blue), (red, blue, red), (red, green, red), (green, blue, green), (green, red, green)}

Finish



Domain(SA, NSW) =
{(blue, green), (blue, red),
(green, blue), (green, red),
(red, blue), (red, green)}

Can identify all settings of SA, V, NSW for which there is guaranteed to be a consistent setting of the remaining variables.

Q: How do we get the settings of the other variables?

Variable Elimination

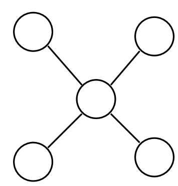
```
Var_elim_CSP_solve (vars, constraints)
Q = queue of all variables
i = length(vars)+1
While not(empty(Q))
    X = pop(Q)
    Xi = merge(X, neighbors(X))
    Simplify Xi
    remove_from_Q(Q, neighbors(X))
    add_to_Q(Q, Xi)
    i=i+1
```

Note: Merge operation can be tricky to implement, depending upon constraint language.

Variable Elimination Issues

- How expensive is this?
- Is it sensitive to elimination ordering?

Variable Elimination Ordering



Is it better to start at the edges and work in, or at the center and work out?

Variable Elimination Facts

- You can figure out the cost of a particular elimination ordering without actually constructing the tables
- Finding optimal elimination ordering is NP hard
- Good heuristics for finding near optimal orderings
- Another structural complexity measure
- Investment in finding good ordering can be amortized

Structural Complexity

- Structural complexity is a somewhat different view of computational complexity: depends upon problem features, not problem class
- For many problems structural complexity is quite manageable
- Idea: Why not convert other NP-hard problems to CSPs and use structural complexity measures, CSP algorithms to solve?

$$2^{poly(k)} >> 2^k$$

CSP Summary

- CSPs are a specialized language for describing certain types of decision problems
- We can formulate special heuristics and methods for problems that can be described in this language
- In general, CSPs are NP hard
- We can use structural measures of complexity to figure out which ones are really hard