Another Look at Binary Search

- **Magic!**
  - Has been used as the basis for “magical” tricks

- **Find telephone number** *(without computer)* **in seconds**
  - If that isn’t magic, what is?

- **There are less than** $10^{80}$ **atoms in the universe.**
  - If ordered, how long to locate a particular one?

- **Demo!**

- **What is the big-Oh?**
Analysis: Algorithms and Data Structures

- We need a vocabulary to discuss performance and to reason about alternative algorithms and implementations
  - It’s faster! It’s more elegant! It’s safer! It’s cooler!

- We need empirical tests and analytical/mathematical tools
  - Given two methods, which is better? Run them to check.
    - 30 seconds vs. 3 seconds, easy. 5 hours vs. 2 minutes, harder
    - What if it takes two weeks to implement the methods?
  - Use mathematics to analyze the algorithm,
  - The implementation is another matter, cache, compiler optimizations, OS, memory,...
Recursion and recurrences

- **Why are some functions written recursively?**
  - Simpler to understand, but …
  - Mt. Everest reasons

- **Are there reasons to prefer iteration?**
  - Better optimizer: programmer/scientist v. compiler
  - Six of one? Or serious differences
    - “One person’s meat is another person’s poison”
    - “To each his own”, “Chacun a son gout”, …

- **Complexity (big-Oh) for iterative and recursive functions**
  - How to determine, estimate, intuit
What’s the complexity of this code?

```
// first and last are integer indexes, list is List
int lastIndex = first;
Object pivot = list.get(first);
for(int k=first+1; k <= last; k++){
    Comparable ko = (Comparable) list.get(k);
    if (ko.compareTo(pivot) <= 0){
        lastIndex++;
        Collections.swap(list,lastIndex,k);
    }
}
```

- What is big-Oh cost of a loop that visits \( n \) elements of a vector?
  - Depends on loop body, if body \( \mathcal{O}(1) \) then …
  - If body is \( \mathcal{O}(n) \) then …
  - If body is \( \mathcal{O}(k) \) for \( k \) in \([0..n)\) then …
private Object findHelper(ArrayList list, int first, int last, int kindex) {
    int lastIndex = first;
    Object pivot = list.get(first);
    for (int k = first + 1; k <= last; k++) {
        Comparable ko = (Comparable) list.get(k);
        if (ko.compareTo(pivot) <= 0) {
            lastIndex++;
            Collections.swap(list, lastIndex, k);
        }
    }
    Collections.swap(list, lastIndex, first);
    if (lastIndex == kindex) return list.get(lastIndex);
    if (kindex < lastIndex) return findHelper(list, first, lastIndex - 1, kindex);
    return findHelper(list, lastIndex + 1, last, kindex);
}
Different measures of complexity

- **Worst case**
  - Gives a good upper-bound on behavior
  - Never get worse than this
  - Drawbacks?

- **Average case**
  - What does average mean?
  - Averaged over all inputs? Assuming uniformly distributed random data?
  - Drawbacks?

- **Best case**
  - Linear search, useful?
Multiplying and adding big-Oh

- Suppose we do a linear search then we do another one
  - What is the complexity?
  - If we do 100 linear searches?
  - If we do n searches on a vector of size n?

- What if we do binary search followed by linear search?
  - What are big-Oh complexities? Sum?
  - What about 50 binary searches? What about n searches?

- What is the number of elements in the list (1,2,2,3,3,3)?
  - What about (1,2,2, ..., n,n,...,n)?
  - How can we reason about this?
Helpful formulae

- **We always mean base 2 unless otherwise stated**
  - What is log(1024)?
  - $\log(xy) = \log(x) + \log(y)$
  - $\log(x^n) = n \log(x)$
  - $\log(2^n) = n$
  - $2^{(\log n)} = n$

- **Sums (also, use sigma notation when possible)**
  - $1 + 2 + 4 + 8 + \ldots + 2^k = 2^{k+1} - 1 = \sum_{i=0}^{k} 2^i$
  - $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2} = \sum_{i=1}^{n} i$
  - $a + ar + ar^2 + \ldots + ar^{n-1} = a(r^n - 1)/(r-1) = \sum_{i=0}^{n-1} ar^i$
Recursion Review

- **Recursive functions have two key attributes**
  - There is a *base case*, sometimes called the *exit case*, which does **not** make a recursive call
  - All other cases make recursive call(s), the results of these calls are used to return a value when necessary
    - Ensure that every sequence of calls reaches base case
    - Some measure decreases/moves towards base case
    - “Measure” can be tricky, but usually it’s straightforward

- **Example: sequential search in an ArrayList**
  - If first element is search key, done and return
  - Otherwise look in the “rest of the list”
  - How can we recurse on “rest of list”?
Sequential search revisited

- What is complexity of sequential search? Of code below?

```java
boolean search(ArrayList list, int first, Object target) {
    if (first >= list.size()) return false;
    else if (list.get(first).equals(target))
        return true;
    else return search(list, first+1, target);
}
```

- Why are there three parameters? Same name good idea?

```java
boolean search(ArrayList list, Object target){
    return search(list, 0, target);
}
```
Why we study recurrences/complexity?

- Tools to analyze algorithms
- *Machine-independent* measuring methods
- Familiarity with good data structures/algorithms

- What is CS person: programmer, scientist, engineer?
  
  *scientists build to learn, engineers learn to build*

- Mathematics is a notation that helps in thinking, discussion, programming
Recurrences

- **Summing Numbers**

```c
int sum(int n) {
    if (0 == n) return 0;
    else return n + sum(n-1);
}
```

- **What is complexity? justification?**
- **T(n) = time to compute sum for n**

\[
T(n) = T(n-1) + 1 \\
T(0) = 1
\]

- **instead of 1, use O(1) for constant time**
  - independent of n, the measure of problem size
Solving recurrence relations

- plug, simplify, reduce, guess, verify?

\[
\begin{align*}
T(n) &= T(n-1) + 1 \\
T(0) &= 1 \\
T(n-1) &= T(n-1-1) + 1 \\
T(n) &= [T(n-2) + 1] + 1 = T(n-2)+2 \\
T(n-2) &= T(n-2-1) + 1 \\
T(n) &= [(T(n-3) + 1) + 1] + 1 = T(n-3)+3
\end{align*}
\]

\[
T(n) = T(n-k) + k \quad \text{find the pattern!}
\]

Now, let \( k=n \), then \( T(n) = T(0)+n = 1+n \)

- get to base case, solve the recurrence: \( O(n) \)
What is complexity of $Build$? (what does it do?)

```java
ArrayList build(int n)
{
    if (0 == n) return new ArrayList(); // empty
    ArrayList list = build(n-1);
    for(int k=0;k < n; k++) {
        list.add(new Integer(n));
    }
    return list;
}
```

Write an expression for $T(n)$ and for $T(0)$, solve.
Recognizing Recurrences

- Solve once, re-use in new contexts
  - T must be explicitly identified
  - n must be some measure of size of input/parameter
    - T(n) is the time for quicksort to run on an n-element vector

\[
\begin{align*}
T(n) &= T(n/2) + O(1) & \text{binary search} & \mathcal{O}(\log n) \\
T(n) &= T(n-1) + O(1) & \text{sequential search} & \mathcal{O}(n) \\
T(n) &= 2T(n/2) + O(1) & \text{tree traversal} & \mathcal{O}(n) \\
T(n) &= 2T(n/2) + O(n) & \text{quicksort} & \mathcal{O}(n \log n) \\
T(n) &= T(n-1) + O(n) & \text{selection sort} & \mathcal{O}(n^2)
\end{align*}
\]

- Remember the algorithm, re-derive complexity