Consider the property: Replace one a with b or \( R_{awb} \) for short. The property \( R_{awb} \) applied to a language \( L \) replaces one a in each string with a b. If a string does not have an a, then the string is not in \( R_{awb}(L) \).

Prove that if \( L \) is regular, prove \( R_{awb}(L) \) is regular.

Given a DFA \( M \) for \( L \), construct a NFA \( M' \) from \( M \) as follows:

First create two copies of \( M \). Start in the first copy. For every \( a \) arc in the first copy, add an additional arc from that state to the corresponding state in the second copy labeled b. All final states in the first copy are no longer final. The initial state in the second copy is no longer an initial state.

![Diagram of two copies of M with arcs added between corresponding states]

Note that \( M' \) could be an NFA because states in the first copy of \( M \) that have an a arc might also have had a b arc out of this state in the original \( M \). We added a b arc out of this state making a possibility of two b arcs now out of this state. Still, \( L(M') \) is regular because we can always convert a NFA into an equivalent DFA and it is well known that if there exists a DFA for \( L \), \( L \) is regular.

However, we cannot jump to the conclusion that \( R_{awb}(L) \) is regular because \( L(M') = R_{awb}(L(M)) \) remains to be proved. That is, we need to argue why \( M' \) works, that it accepts strings it should and doesn’t accept strings not in the language.

Claim: \( L(M') = R_{awb}(L(M)) \).

Proof:

\[ w \in L(M') \]

\[ \iff w \text{ can be expressed as } ubv \text{ where } u, v \in \sum^* \text{ and } b \text{ is the label of the arc that leaves from one state in the first copy to another state in the second copy. (This is because all the final states are located in the second copy)} \]

\[ \iff \delta^*(\delta^*(q_0, ub), v) \in F' \text{ in } M' \]
\[ \Leftrightarrow \delta^*(\delta^*(q_0, ua), v) \in F \text{ in } M \]
\[ \Leftrightarrow uav \in L(M) \]
\[ \Leftrightarrow ubv \in R1awb(L(M)) \text{ because we can get } ubv \text{ from } uav \text{ by replacing an } a \text{ with a } b. \]
\[ \Leftrightarrow w \in R1awb(L(M)) \]