Section: Transforming grammars  
(Ch. 6)

Methods for Transforming Grammars

We will consider CFL without $\lambda$. It would be easy to add $\lambda$ to any grammar by adding a new start symbol $S_0$,

$$S_0 \rightarrow S \mid \lambda$$
Theorem (Substitution) Let G be a CFG. Suppose G contains

\[ A \rightarrow x_1Bx_2 \]

where A and B are different variables, and B has the productions

\[ B \rightarrow y_1 | y_2 | \ldots | y_n \]

Then can construct G’ from G by deleting

\[ A \rightarrow x_1Bx_2 \]

from P and adding to it

\[ A \rightarrow x_1y_1x_2 | x_1y_2x_2 | \ldots | x_1y_nx_2 \]

Then, \( L(G) = L(G') \).
Example:

\[
S \rightarrow aBa \quad \text{becomes} \\
B \rightarrow aS \mid a
\]

Definition: A production of the form \( A \rightarrow Ax, A \in V, x \in (V \cup T)^* \) is \textit{left recursive}. 
Example Previous expression grammar was left recursive.

\[
E \rightarrow E + T \mid T \\
T \rightarrow T * F \mid F \\
F \rightarrow I \mid (E) \\
I \rightarrow a \mid b
\]

Derivation of \(a + b + a + a\) is:

\[
E \Rightarrow E + T \Rightarrow E + T + T \Rightarrow E + T + T + T \Rightarrow a + T + T + T
\]
Theorem (Removing Left recursion)
Let $G=(V,T,S,P)$ be a CFG. Divide productions for variable $A$ into left-recursive and non left-recursive productions:

$$
A \rightarrow A.x_1 \mid A.x_2 \mid \ldots \mid A.x_n \\
A \rightarrow y_1 | y_2 | \ldots | y_m
$$

where $x_i, y_i$ are in $(V \cup T)^*$.

Then $G'=(V\cup \{Z\}, T, S, P')$ and $P'$ replaces rules of form above by

$$
A \rightarrow y_i | y_i Z, \ i=1,2,\ldots,m \\
Z \rightarrow x_i | x_i Z, \ i=1,2,\ldots,n
$$
Example:

\[ E \rightarrow E + T | T \] becomes

\[ T \rightarrow T \ast F | F \] becomes

Now, Derivation of \( a+b+a+a \) is:
Useless productions

S → aB | bA
A → aA
B → Sa
C → cBc | a

What can you say about this grammar?

Theorem (useless productions) Let G be a CFG. Then ∃ G’ that does not contain any useless variables or productions s.t. L(G)=L(G’).
To Remove Useless Productions:

Let $G = (V, T, S, P)$.

I. Compute $V_1 = \{\text{Variables that can derive strings of terminals}\}$

1. $V_1 = \emptyset$

2. Repeat until no more variables added
   - For every $A \in V$ with $A \rightarrow x_1 x_2 \ldots x_n$, $x_i \in (T* \cup V_1)$, add $A$ to $V_1$

3. $P_1 = \text{all productions in } P \text{ with symbols in } (V_1 \cup T)^*$

Then $G_1 = (V_1, T, S, P_1)$ has no variables that can’t derive strings.
II. Draw Variable Dependency Graph

For $A \rightarrow xBy$, draw $A \rightarrow B$.

Remove productions for $V$ if there is no path from $S$ to $V$ in the dependency graph. Resulting Grammar $G'$ is s.t. $L(G) = L(G')$ and $G'$ has no useless productions.
Example:

\[ S \rightarrow aB \mid bA \]
\[ A \rightarrow aA \]
\[ B \rightarrow Sa \mid b \]
\[ C \rightarrow cBc \mid a \]
\[ D \rightarrow bCb \]
\[ E \rightarrow Aa \mid b \]
Theorem (remove \( \lambda \) productions) Let \( G \) be a CFG with \( \lambda \) not in \( L(G) \). Then \( \exists \) a CFG \( G' \) having no \( \lambda \)-productions s.t. \( L(G) = L(G') \).

To Remove \( \lambda \)-productions

1. Let \( V_n = \{ A \mid \exists \) production \( A \rightarrow \lambda \} \)
2. Repeat until no more additions
   - if \( B \rightarrow A_1 A_2 \ldots A_m \) and \( A_i \in V_n \) for all \( i \), then put \( B \) in \( V_n \)
3. Construct \( G' \) with productions \( P' \) s.t.
   - if \( A \rightarrow x_1 x_2 \ldots x_m \in P, m \geq 1 \), then put all productions formed when \( x_j \) is replaced by \( \lambda \) (for all \( x_j \in V_n \)) s.t. \( |\text{rhs}| \geq 1 \) into \( P' \).
Example:

S → Ab
A → BCB | Aa
B → b | λ
C → cC | λ
Definition Unit Production

A → B

where A,B ∈ V.

Consider removing unit productions:

Suppose we have

A → B becomes
B → a | ab

But what if we have

A → B becomes
B → C
C → A
Theorem (Remove unit productions)
Let \( G = (V, T, S, P) \) be a CFG without \( \lambda \)-productions. Then \( \exists \) CFG \( G’ = (V’, T’, S, P’) \) that does not have any unit-productions and \( L(G) = L(G’) \).

To Remove Unit Productions:

1. Find for each \( A \), all \( B \) s.t. \( A \Rightarrow^* B \) (Draw a dependency graph)
2. Construct \( G’ = (V’, T’, S, P’) \) by
   
   (a) Put all non-unit productions in \( P’ \)
   (b) For all \( A \Rightarrow^* B \) s.t. \( B \rightarrow y_1 | y_2 | \ldots y_n \in P’ \), put \( A \rightarrow y_1 | y_2 | \ldots y_n \in P’ \)
Example:

\[ S \rightarrow AB \]
\[ A \rightarrow B \]
\[ B \rightarrow C \mid Bb \]
\[ C \rightarrow A \mid c \mid Da \]
\[ D \rightarrow A \]
Theorem Let L be a CFL that does not contain $\lambda$. Then $\exists$ a CFG for L that does not have any useless productions, $\lambda$-productions, or unit-productions.

Proof

1. Remove $\lambda$-productions
2. Remove unit-productions
3. Remove useless productions

Note order is very important. Removing $\lambda$-productions can create unit-productions! QED.
Definition: A CFG is in Chomsky Normal Form (CNF) if all productions are of the form

\[ A \rightarrow BC \quad \text{or} \quad A \rightarrow a \]

where \( A, B, C \in V \) and \( a \in T \).

Theorem: Any CFG \( G \) with \( \lambda \) not in \( L(G) \) has an equivalent grammar in CNF.

Proof:

1. Remove \( \lambda \)-productions, unit productions, and useless productions.

2. For every rhs of length > 1, replace each terminal \( x_i \) by a new variable \( C_j \) and add the production \( C_j \rightarrow x_i \).

3. Replace every rhs of length > 2 by a series of productions, each with rhs of length 2. QED.
Example:

\[ S \rightarrow CB\text{cd} \]
\[ B \rightarrow b \]
\[ C \rightarrow Cc \mid e \]
Definition: A CFG is in Greibach normal form (GNF) if all productions have the form

\[ A \rightarrow ax \]

where \( a \in T \) and \( x \in V^* \)

Theorem For every CFG \( G \) with \( \lambda \) not in \( L(G) \), \( \exists \) a grammar in GNF.

Proof:

1. Rewrite grammar in CNF.
2. Relabel Variables \( A_1, A_2, \ldots A_n \)
3. Eliminate left recursion and use substitution to get all productions into the form:

\[
A_i \rightarrow A_j x_j, \ j > i \\
Z_i \rightarrow A_j x_j, \ j \leq n \\
A_i \rightarrow ax_i
\]

where \(a \in T, \ x_i \in V^*,\) and \(Z_i\) are new variables introduced for left recursion.

4. All productions with \(A_n\) are in the correct form, \(A_n \rightarrow ax_n.\) Use these productions as substitutions to get \(A_{n-1}\) productions in the correct form. Repeat with \(A_{n-2}, A_{n-3},\) etc until all productions are in the correct form.