Read Section 12.1.

**Computability** A function $f$ with domain $D$ is *computable* if there exists some TM $M$ such that $M$ computes $f$ for all values in its domain.

**Decidability** A problem is *decidable* if there exists a TM that can answer yes or no to every statement in the domain of the problem.

**The Halting Problem**

Domain: set of all TMs and all strings $w$.

Question: Given coding of $M$ and $w$, does $M$ halt on $w$? (yes or no)

**Theorem** The halting problem is undecidable.

**Proof:** (by contradiction)

- Assume there is a TM $H$ (or algorithm) that solves this problem.
  - TM $H$ has 2 final states, $q_y$ represents yes and $q_n$ represents no.
  - TM $H$ has input the coding of TM $M$ (denoted $w_M$) and input string $w$ and ends in state $q_y$ (yes) if $M$ halts on $w$ and ends in state $q_n$ (no) if $M$ doesn’t halt on $w$.

$$H(w_M, w) = \begin{cases} 
  \text{(yes) halts in } q_y & \text{if } M \text{ halts on } w \\
  \text{(no) halts in } q_n & \text{if } M \text{ doesn’t halt on } w
\end{cases}$$

TM $H$ always halts in a final state.

Construct TM $H'$ from $H$ such that $H'$ halts if $H$ ends in state $q_n$ and $H'$ doesn’t halt if $H$ ends in state $q_y$.

$$H'(w_M, w) = \begin{cases} 
  \text{halts} & \text{if } M \text{ doesn’t halt on } w \\
  \text{doesn’t halt} & \text{if } M \text{ halts on } w
\end{cases}$$
Construct TM $\hat{H}$ from $H'$ such that $\hat{H}$ makes a copy of $w_M$ and then behaves like $H'$. (simulates TM $M$ on the input string that is the encoding of TM $M$, applies $M_w$ to $M_w$).

So $\hat{H}(w_M)$ runs $H'(w_M, w_M)$

$$\hat{H}(w_M) = \begin{cases} \text{halts} & \text{if } M \text{ doesn’t halt on } w_M \\ \text{doesn’t halt} & \text{if } M \text{ halts on } w_M \end{cases}$$

Note that $\hat{H}$ is a TM.

There is some encoding of it, say $\hat{w}_\hat{H}$.

What happens if we run $\hat{H}$ with input $\hat{w}_\hat{H}$?

**Theorem** If the halting problem were decidable, then every recursively enumerable language would be recursive. Thus, the halting problem is undecidable.

- **Proof**: Let $L$ be an RE language over $\Sigma$.
  
  Let $M$ be the TM such that $L = L(M)$.
  
  Let $H$ be the TM that solves the halting problem.
A problem A is *reduced* to problem B if the decidability of B follows from the decidability of A. Then if we know B is undecidable, then A must be undecidable.

**State-entry problem** Given TM \( M = (Q, \Sigma, \Gamma, \delta, q_0, B, F) \), state \( q \in Q \), and string \( w \in \Sigma^* \), is state \( q \) ever entered when \( M \) is applied to \( w \)?

This is an undecidable problem!

- **Proof:** We will reduce this problem to the halting problem.

  Suppose we have a TM \( E \) to solve the state-entry problem.

  TM \( E \) takes as input the coding of a TM \( M \) (denoted by \( w_M \)), a string \( w \) and a state \( q \). TM \( E \) answers *yes* if state \( q \) is entered and *no* if state \( q \) is not entered.

  Construct TM \( E' \) which does the following. On input \( w_M \) and \( w \) \( E' \) first examines the transition functions of \( M \). Whenever \( \delta \) is not defined for some state \( q_i \) and symbol \( a \) add the transition \( \delta(q_i, a) = (q, a, R) \). Let this new state \( q \) be the only final state. Let \( M' \) be the modified TM. Next, simulate TM \( E \) on input \( w_M', w \) and \( q \).

\[
E'(w_M, w) = \begin{cases} 
\text{M halts on } w & \text{if } M' \text{ enters state } q \\
\text{M doesn’t halt on } w & \text{if } M' \text{ doesn’t enter state } q 
\end{cases}
\]

TM \( E' \) determines if \( M \) halts on \( w \). If \( M \) halts on \( w \) then TM \( E' \) will enter state \( q \) in \( M' \) and answer *yes*. If \( M \) doesn’t halt on \( w \) then TM \( E' \) will not enter state \( q \), so it will answer *no*. Since the state-entry problem is decidable, \( E \) always gives an answer *yes* or *no*.

But the halting problem is undecidable. Contradiction! Thus, the state-entry problem must be undecidable. QED.

There are some more examples of undecidability in section 12.1.