Section: Pushdown Automata

Ch. 7 - Pushdown Automata

A DFA = \((Q, \Sigma, \delta, q_0, F')\)

![Diagram of a DFA with an input tape, tape head, and current state]
Modify DFA by adding a stack. New machine is called Pushdown Automata (PDA).

```
input tape

a a a a b b

tape head

head moves →

current state

0 1

stack

a
a
Z
```
Definition: Nondeterministic PDA (NPDA) is defined by

\[ M = (Q, \Sigma, \Gamma, \delta, q_0, z, F) \]

where

- \( Q \) is finite set of states
- \( \Sigma \) is tape (input) alphabet
- \( \Gamma \) is stack alphabet
- \( q_0 \) is initial state
- \( z \) - start stack symbol, \( z \in \Gamma \)
- \( F \subseteq Q \) is set of final states.

\[ \delta : Q \times (\Sigma \cup \{ \lambda \}) \times \Gamma \rightarrow \text{finite subsets of } Q \times \Gamma^* \]
Example of transitions

\[ \delta(q_1, a, b) = \{(q_3, b), (q_4, ab), (q_6, \lambda)\} \]

The diagram for the above transitions is:
Instantaneous Description:

\((q,w,u)\)

Description of a Move:

\((q_1,aw,bx) \vdash (q_2,w,yx)\)

iff

Definition Let \(M=(Q,\Sigma,\Gamma,\delta,q_0,z,F)\) be a NPDA. \(L(M) = \{w \in \Sigma^* \mid (q_0,w,z) \vdash^*(p,\lambda,u), p \in F, u \in \Gamma^*\}\). The NPDA accepts all strings that start in \(q_0\) and end in a final state.
Example: \( L = \{ a^n b^n | n \geq 0 \} \), \( \Sigma = \{ a, b \} \),
\( \Gamma = \{ z, a \} \)
Another Definition for Language Acceptance

NPDA $M$ accepts $L(M)$ by empty stack:

$$L(M) = \{ w \in \Sigma^* \mid (q_0, w, z)^* \vdash (p, \lambda, \lambda) \}$$
Example: \( L = \{a^n b^m c^{n+m} | n, m > 0 \} \), 
\( \Sigma = \{a, b, c\} \), \( \Gamma = \{0, z\} \)
Example: $L = \{ww^R| w \in \Sigma^+\}$, $\Sigma = \{a, b\}$, $\Gamma = \{z, a, b\}$
Example: \( L = \{ww | w \in \Sigma^*\} \), \( \Sigma = \{a, b\} \)

Examples for you to try on your own: (solutions are at the end of the handout).

- \( L = \{a^n b^m | m > n, m, n > 0\} \), \( \Sigma = \{a, b\} \), \( \Gamma = \{z, a\} \)
- \( L = \{a^n b^{n+m} c^m | n, m > 0\} \), \( \Sigma = \{a, b, c\} \)
- \( L = \{a^n b^{2n} | n > 0\} \), \( \Sigma = \{a, b\} \)
Definition: A PDA

$M = (Q, \Sigma, \Gamma, \delta, q_0, z, F)$ is deterministic if for every $q \in Q$, $a \in \Sigma \cup \{\lambda\}$, $b \in \Gamma$

1. $\delta(q, a, b)$ contains at most 1 element
2. if $\delta(q, \lambda, b) \neq \emptyset$ then $\delta(q, c, b) = \emptyset$ for all $c \in \Sigma$

Definition: $L$ is DCFL iff $\exists$ DPDA $M$ s.t. $L = L(M)$. 
Examples:

1. Previous pda for \( \{ a^n b^n | n \geq 0 \} \) is deterministic?

2. Previous pda for \( \{ a^n b^m c^{n+m} | n, m > 0 \} \) is deterministic?

3. Previous pda for \( \{ w w^R | w \in \Sigma^+ \}, \Sigma = \{ a, b \} \) is deterministic?
Example: $L = \{a^n b^m | m > n, m, n > 0\}$, 
$\Sigma = \{a, b\}$, $\Gamma = \{z, a\}$

Example: $L = \{a^n b^{n+m} c^m | n, m > 0\}$, 
$\Sigma = \{a, b, c\}$

Example: $L = \{a^n b^{2n} | n > 0\}$, $\Sigma = \{a, b\}$
Chapter 7.2

Theorem Given NPDA $M$ that accepts by final state, $\exists$ NPDA $M'$ that accepts by empty stack s.t. $L(M) = L(M')$.

- Proof (sketch)
  
  $M = (Q, \Sigma, \Gamma, \delta, q_0, z, F)$

  Construct $M' = (Q', \Sigma', \Gamma', \delta', q_s, z', F')$
Theorem Given NPDA $M$ that accepts by empty stack, $\exists$ NPDA $M'$ that accepts by final state.

- Proof: (sketch)
  
  $M = (Q, \Sigma, \Gamma, \delta, q_0, z, F)$

  Construct $M' = (Q', \Sigma, \Gamma', \delta', q_s, z', F')$
Theorem For any CFL L not containing $\lambda$, $\exists$ an NPDA M s.t. $L=L(M)$.

- Proof (sketch)
  
  Given ($\lambda$-free) CFL L.
  
  $\Rightarrow \exists$ CFG G such that $L=L(G)$.
  
  $\Rightarrow \exists$ G’ in GNF, s.t. $L(G)=L(G’)$.
  
  G’=(V,T,S,P). All productions in P are of the form:
Example: Let $G' = (V, T, S, P)$, $P =$

$$
S \rightarrow aSA \mid aAA \mid b \\
A \rightarrow bBBB \\
B \rightarrow b
$$
Theorem Given a NPDA M, ∃ a NPDA M’ s.t. all transitions have the form \( \delta(q_i,a,A) = \{c_1, c_2, \ldots c_n\} \) where

\[
c_i = (q_j, \lambda)
\]

or \( c_i = (q_j, BC) \)

Each move either increases or decreases stack contents by a single symbol.

• Proof (sketch)
Theorem If \(L = L(M)\) for some NPDA \(M\), then \(L\) is a CFL.

• **Proof**: Given NPDA \(M\).
  First, construct an equivalent NPDA \(M\) that will be easier to work with. Construct \(M'\) such that
  1. accepts if stack is empty
  2. each move increases or decreases stack content by a single symbol.
     (can only push 2 variables or no variables with each transition)

\[
M' = (Q, \Sigma, \Gamma, \delta, q_0, z, F)
\]

Construct \(G = (V, \Sigma, S, P)\) where

\[
V = \{(q_i c q_j) | q_i, q_j \in Q, c \in \Gamma\}
\]

**Goal**: \((q_0 z q_f)\) which will be the start symbol in the grammar.
Example:

\[ L(M) = \{ aa^* b \}, \quad M = (Q, \Sigma, \Gamma, \delta, q_0, z, F), \]
\[ Q = \{ q_0, q_1, q_2, q_3 \}, \]
\[ \Sigma = \{ a, b \}, \Gamma = \{ A, z \}, F = \{ \}. \]
Construct the grammar $G = (V, T, S, P)$,

$V = \{(q_0\Delta q_0), (q_0\varepsilon q_0), (q_0\Delta q_1), (q_0\varepsilon q_1), \ldots\}$

$T = \Sigma$

$S = (q_0\varepsilon q_2)$

$P =$
From transition 1 \((q_0Aq_1) \rightarrow b\)

From transition 2 \((q_1zq_2) \rightarrow \lambda\)

From transition 3 \((q_0Aq_3) \rightarrow a\)

From transition 4 \((q_0zq) \rightarrow a(q_0Aq_0)(q_0zq) | a(q_0Aq_1)(q_1zq) | a(q_0Aq_2)(q_2zq) | a(q_0Aq_3)(q_3zq)\)

\((q_0zq_1) \rightarrow a(q_0Aq_0)(q_0zq_1) | a(q_0Aq_1)(q_1zq_1) | a(q_0Aq_2)(q_2zq_1) | a(q_0Aq_3)(q_3zq_1)\)

\((q_0zq_2) \rightarrow a(q_0Aq_0)(q_0zq_2) | a(q_0Aq_1)(q_1zq_2) | a(q_0Aq_2)(q_2zq_2) | a(q_0Aq_3)(q_3zq_2)\)

\((q_0zq_3) \rightarrow a(q_0Aq_0)(q_0zq_3) | a(q_0Aq_1)(q_1zq_3) | a(q_0Aq_2)(q_2zq_3) | a(q_0Aq_3)(q_3zq_3)\)
From transition 5 

\[(q_3 z q_0) \rightarrow (q_0 A q_0)(q_0 z q_0)\]
\[(q_0 A q_1)(q_1 z q_0)\]
\[(q_0 A q_2)(q_2 z q_0)\]
\[(q_0 A q_3)(q_3 z q_0)\]

\[(q_3 z q_1) \rightarrow (q_0 A q_0)(q_0 z q_1)\]
\[(q_0 A q_1)(q_1 z q_1)\]
\[(q_0 A q_2)(q_2 z q_1)\]
\[(q_0 A q_3)(q_3 z q_1)\]

\[(q_3 z q_2) \rightarrow (q_0 A q_0)(q_0 z q_2)\]
\[(q_0 A q_1)(q_1 z q_2)\]
\[(q_0 A q_2)(q_2 z q_2)\]
\[(q_0 A q_3)(q_3 z q_2)\]

\[(q_3 z q_3) \rightarrow (q_0 A q_0)(q_0 z q_3)\]
\[(q_0 A q_1)(q_1 z q_3)\]
\[(q_0 A q_2)(q_2 z q_3)\]
\[(q_0 A q_3)(q_3 z q_3)\]
Recognizing aaab in M:

\[(q_0, aaab, z) \vdash (q_0, aab, Az)\]
\[\vdash (q_3, ab, z)\]
\[\vdash (q_0, ab, Az)\]
\[\vdash (q_3, b, z)\]
\[\vdash (q_0, b, Az)\]
\[\vdash (q_1, \lambda, z)\]
\[\vdash (q_2, \lambda, \lambda)\]

Derivation of string aaab in G:

\[(q_0 z q_2) \Rightarrow a(q_0 Aq_3)(q_3 z q_2)\]
\[\Rightarrow aa(q_3 z q_2)\]
\[\Rightarrow aa(q_0 Aq_3)(q_3 z q_2)\]
\[\Rightarrow aaa(q_3 z q_2)\]
\[\Rightarrow aaa(q_0 Aq_1)(q_1 z q_2)\]
\[\Rightarrow aaab(q_1 z q_2)\]
\[\Rightarrow aaab\]
Chapter 7.3

Definition: A PDA

$M = (Q, \Sigma, \Gamma, \delta, q_0, z, F)$ is deterministic if for every $q \in Q$, $a \in \Sigma \cup \{\lambda\}$, $b \in \Gamma$

1. $\delta(q, a, b)$ contains at most 1 element
2. if $\delta(q, \lambda, b) \neq \emptyset$ then $\delta(q, c, b) = \emptyset$ for all $c \in \Sigma$

Definition: $L$ is DCFL iff $\exists$ DPDA $M$ s.t. $L = L(M)$. 
Examples:

1. Previous pda for \( \{a^n b^n | n \geq 0 \} \) is deterministic.

2. Previous pda for \( \{a^n b^m c^{n+m} | n, m > 0 \} \) is deterministic.

3. Previous pda for \( \{ww^R | w \in \Sigma^+ \}, \Sigma = \{a, b\} \) is nondeterministic.

Note: There are CFL’s that are not deterministic.

\[ L = \{a^n b^n | n \geq 1\} \cup \{a^n b^{2n} | n \geq 1\} \] is a CFL and not a DCFL.

- Proof:
  \[ L = \{a^n b^n : n \geq 1\} \cup \{a^n b^{2n} : n \geq 1\} \]
  It is easy to construct a NPDA for \( \{a^n b^n : n \geq 1\} \) and a NPDA for \( \{a^n b^{2n} : n \geq 1\} \). These two can be joined together by a new start state
and $\lambda$-transitions to create a NPDA for $L$. Thus, $L$ is CFL.

Now show $L$ is not a DCFL. Assume that there is a deterministic PDA $M$ such that $L = L(M)$. We will construct a PDA that recognizes a language that is not a CFL and derive a contradiction.

Construct a PDA $M'$ as follows:

1. Create two copies of $M$: $M_1$ and $M_2$. The same state in $M_1$ and $M_2$ are called cousins.

2. Remove accept status from accept states in $M_1$, remove initial status from initial state in $M_2$. In our new PDA, we will start in $M_1$ and accept in $M_2$.

3. Outgoing arcs from old accept states in $M_1$, change to end up in the cousin of its destination in
$M_2$. This joins $M_1$ and $M_2$ into one PDA. There must be an outgoing arc since you must recognize both $a^n b^n$ and $a^n b^{2n}$. After reading $n$ $b$’s, must accept if no more $b$’s and continue if there are more $b$’s.

4. Modify all transitions that read a $b$ and have their destinations in $M_2$ to read a $c$.

This is the construction of our new PDA.

When we read $a^n b^n$ and end up in an old accept state in $M_1$, then we will transfer to $M_2$ and read the rest of $a^n b^{2n}$. Only the $b$’s in $M_2$ have been replaced by $c$’s, so the new machine accepts $a^n b^n c^n$.

The language accepted by our new PDA is $a^n b^n c^n$. But this is not a CFL. Contradiction! Thus there is
no deterministic PDA $M$ such that $L(M) = L$. Q.E.D.