Section: Properties of Regular Languages

Example

$L = \{ a^n b a^n \mid n > 0 \}$

Closure Properties

A set is closed over an operation if

$L_1, L_2 \in \text{class}$
$L_1 \text{ op } L_2 = L_3$
$
\Rightarrow L_3 \in \text{class}$
$L_1 = \{ x \mid x \text{ is a positive even integer} \}$

$L$ is closed under

- addition?
- multiplication?
- subtraction?
- division?

Closure of Regular Languages

Theorem 4.1 If $L_1$ and $L_2$ are regular languages, then

- $L_1 \cup L_2$
- $L_1 \cap L_2$
- $L_1L_2$
- $\overline{L_1}$
- $L_1^*$

are regular languages.
Proof(sketch)

$L_1$ and $L_2$ are regular languages

$\Rightarrow \exists$ reg. expr. $r_1$ and $r_2$ s.t.

$L_1 = L(r_1)$ and $L_2 = L(r_2)$

$r_1 + r_2$ is r.e. denoting $L_1 \cup L_2$

$\Rightarrow$ closed under union

$r_1r_2$ is r.e. denoting $L_1L_2$

$\Rightarrow$ closed under concatenation

$r_1^*$ is r.e. denoting $L_1^*$

$\Rightarrow$ closed under star-closure
complementation:

$L_1$ is reg. lang.

$\Rightarrow \exists$ DFA $M$ s.t. $L_1 = L(M)$

Construct $M'$ s.t.

- final states in $M$ are nonfinal states in $M'$
- nonfinal states in $M$ are final states in $M'$

$\Rightarrow$ closed under complementation
intersection:

$L_1$ and $L_2$ are reg. lang.

$\Rightarrow \exists$ DFA $M_1$ and $M_2$ s.t.

$L_1 = L(M_1)$ and $L_2 = L(M_2)$

$M_1 = (Q, \Sigma, \delta_1, q_0, F_1)$

$M_2 = (P, \Sigma, \delta_2, p_0, F_2)$

Construct $M' = (Q', \Sigma, \delta', (q_0, p_0), F')$

$Q' = (Q \times P)$

$\delta'$:

$\delta'((q_i, p_j), a) = (q_k, p_l)$ if

$w \in L(M') \iff w \in L_1 \cap L_2$

$\Rightarrow$ closed under intersection
Example:
Regular languages are closed under

- reversal $L^R$
- difference $L_1 - L_2$
- right quotient $L_1 / L_2$
- homomorphism $h(L)$
Right quotient

Def: $L_1/L_2 = \{x | xy \in L_1 \text{ for some } y \in L_2 \}$

Example:

$L_1 = \{a^*b^* \cup b^*a^* \}$
$L_2 = \{b^n | n \text{ is even, } n > 0 \}$
$L_1/L_2 =$
Theorem If $L_1$ and $L_2$ are regular, then $L_1/L_2$ is regular.

Proof (sketch)

$\exists$ DFA $M=\langle Q, \Sigma, \delta, q_0, F \rangle$ s.t. $L_1 = L(M)$.

Construct DFA $M'=\langle Q, \Sigma, \delta, q_0, F' \rangle$

For each state $i$ do

Make $i$ the start state (representing $L'_i$) if $L'_i \cap L_2 \neq \emptyset$ then put $q_i$ in $F'$ in $M'$

QED.
Homomorphism

Def. Let \( \Sigma, \Gamma \) be alphabets. A homomorphism is a function

\[
h: \Sigma \rightarrow \Gamma^*
\]

Example:

\[
\Sigma = \{a, b, c\}, \quad \Gamma = \{0, 1\}
\]

\[
h(a) = 11 \\
h(b) = 00 \\
h(c) = 0
\]

\[
h(bc) =
\]

\[
h(ab^*) =
\]
Questions about regular languages:

L is a regular language.

• Given L, \( \Sigma \), \( w \in \Sigma^* \), is \( w \in L \)?

• Is L empty?

• Is L infinite?

• Does \( L_1 = L_2 \)?
Identifying Nonregular Languages

If a language $L$ is finite, is $L$ regular?

If $L$ is infinite, is $L$ regular?

- $L_1 = \{a^n b^m | n > 0, m > 0\} = aa^*bb^*$
- $L_2 = \{a^n b^n | n > 0\}$
Prove that $L_2 = \{a^n b^n | n > 0\}$ is ?

• Proof:
Pumping Lemma: Let $L$ be an infinite regular language. $\exists$ a constant $m > 0$ such that any $w \in L$ with $|w| \geq m$ can be decomposed into three parts as $w = xyz$ with

$$
|xy| \leq m \\
|y| \geq 1 \\
xy^iz \in L \text{ for all } i \geq 0
$$
To Use the Pumping Lemma to prove L is not regular:

• Proof by Contradiction.
  Assume L is regular.
  ⇒ L satisfies the pumping lemma.
  Choose a long string \( w \) in L, \(|w| \geq m\).
  Show that there is NO division of \( w \) into \( xyz \) (must consider all possible divisions) such that \(|xy| \leq m\), \(|y| \geq 1\) and \( xy^i z \in L \ \forall \ i \geq 0\).
  The pumping lemma does not hold. Contradiction!
  ⇒ L is not regular. QED.
Example $L = \{a^n b^n | n > 0\}$

$L$ is not regular.

• Proof:
  
  Assume $L$ is regular.
  
  $\Rightarrow$ the pumping lemma holds.
  
  Choose $w =$
Example $L = \{a^n b^{n+s} c^s | n, s > 0\}$

$L$ is not regular.

- **Proof:**
  
  Assume $L$ is regular.
  
  $\Rightarrow$ the pumping lemma holds.
  
  Choose $w =$
  
  So the partition is:
Example $\Sigma = \{a, b\}$, 
$L = \{w \in \Sigma^* \mid n_a(w) > n_b(w)\}$

$L$ is not regular.

• Proof:
  
  Assume $L$ is regular.
  
  $\Rightarrow$ the pumping lemma holds.
  
  Choose $w =$
  
  So the partition is:
Example \( L = \{a^3b^n c^{n-3} | n > 3\} \)

\( L \) is not regular.
To Use Closure Properties to prove $L$ is not regular:

- Proof Outline:
  Assume $L$ is regular.
  Apply closure properties to $L$ and other regular languages, constructing $L'$ that you know is not regular.
  closure properties $\Rightarrow L'$ is regular.
  Contradiction!
  $L$ is not regular. QED.
Example $L = \{a^3b^nc^{n-3} | n > 3\}$

$L$ is not regular.

- **Proof: (proof by contradiction)**
  
  Assume $L$ is regular.
  
  Define a homomorphism $h : \Sigma \rightarrow \Sigma^*$
  
  $h(a) = a$  $h(b) = a$  $h(c) = b$
  
  $h(L) =$
Example $L = \{a^n b^m a^m | m \geq 0, n \geq 0\}$

$L$ is not regular.

- Proof: (proof by contradiction)
  Assume $L$ is regular.
Example: \( L_1 = \{ a^n b^n a^n | n > 0 \} \)

\( L_1 \) is not regular.