1. Given a finite set of sampled points \((x_j, f(x_j)), j = 0 : n\), of a function, which is not a polynomial of degree less than or equal to 2. Assume without loss of generality that \(x_j < x_{j+1}\). Assume \(h_j = x_j - x_{j-1}\) are all small, \(0 < h_j < h < 1\), and \(n > 10\).

   (a) To approximate \(f''(x)\) at \(x \in [x_1, x_{n-1}]\), which divided difference you recommend to use and why. (10 points)

   (b) Fix a divided difference \(f[x_{k-1}, x_k, x_{k+1}]\), without reference to any of the boundary points. At which point \(x \in [x_{k-1}, x_{k+1}]\), \(f''(x)\) can be approximated the best in terms of an error bound. (10 points)

   (c) How to approximate \(f''(x)\) when \(x\) is close or at the boundary points with the approximation accuracy comparable to the points in interior sub-intervals? (10 points)

   (d) Provide two sets of experimental results based on your claims on the above issues. (10 points)

     - The samples are equally spaced in \(x\) in one experiment and randomly spaced in another, within the same predescribed interval \([x_0, x_n]\).
     - Choose your own test functions, give a brief reason.
     - The test points shall be much denser than the sample points. (Use test points to get the “true” values)

2. Given a finite set of sampled points \((x_j, f(x_j)), j = 0 : n\), of a function, which is not a polynomial of degree less than or equal to 2. Assume that \(x_j = j \times h + x_0\) for a small \(h < 1\).

   (a) Apply Richardson’s method to get an computationally easy method to estimate \(f'''\) in order to sharpen sample-based error estimate. (10 points)

   (b) Apply Richardson’s extrapolation method to get better approximation of \(f''(x)\) using 5 sampling points. (10 points)

   (c) Provide experimental comparison to the previous experiment (problem 1.d) with equally-spaced sampling. (10 points)

   (d) (Optional) Assume there are some ‘missing’ samples on the equally spaced grid. How to use Richardson’s scheme? (extra 10 points)
3. Consider the application of the trapezoidal rule for numerical integration of a periodic function over a finite interval \([a, b]\).

(a) Present a method and its error analysis, based on Taylor’s theorem. (5 points)

(b) Present a method and its error analysis, based on Fourier’s theorem. (5 points)

4. Describe the similarity and difference between Gaussian-Legendre quadrature and Gaussian-Chebyshev quadrature. (10 points)

5. How would you select a quadrature when given a particular function to integrate? (10 points)