1. Given a set of sample points \((x_i, y_i), i = 1 : m\). Provided with a set of basis/feature functions \(b_j(x), j = 1 : n\), which may be continuous, or defined on \(x_i\) only.

   (a) Describe a LS data fitting problem with a linear model. And verify that there always exists a LS solution.

   (b) Provide the condition for unique LS solution and an analytical expression of the unique solution.

   (c) When the uniqueness condition is not met, provide an expression of general solution via a particular LS solution.

2. QR factorizations of a matrix: experimental comparison (preferably matlab code)

   (a) A QR factorization using Householder reflections (successive orthogonal transforms).

   The subroutine for determining the orthogonal transformation is provided.

   (b) Another QR factorization using the Gram-Schmidt procedure (successive orthogonal projections).

   (c) Compare the orthogonal factors, the triangular factors, and the numerical solutions to a LS problem with a linear model. Test with at least two kinds of matrices.

   (d) Add the component of column pivoting in the QR factorization routines. And make the comparison again.

3. Assume that \(p_k(x), k = 0, 1, \cdots\), are orthogonal polynomials with respect to certain inner product and with increasing degree. Verify the following.

   (a) There is a triangular factor between \(\{p_k(x)\}\) and the monomials \(\{x^k\}\).

   (b) There is a three-term recursive relationship among the polynomials \(p_k(x)\). 

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4. Derive the sampling theorem for bandlimited functions by using the inverse Fourier transform and the Fourier series. And verify the following.

(a) The translated sinc functions are orthogonal with respect to the inner product over $L_2$ space. (Defined as $\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)g(x)dx$)

Note also that the translated sinc functions are cardinal on the sample points.

(b) Let $\sigma > 0$ be fixed. Let $P(f) = \text{sinc} * f$. Then, $P$ is a projector in the temporal/spatial space to the space of functions bandlimited by $\sigma$.

(c) (Optional) A bandlimited function is not timelimited.

5. The B-splines. Let $\beta^0(x)$ be the characteristic function of the interval $[-1/2, 1/2]$. Define $\beta^k = \beta^0 * \beta^{k-1}$ as the B-spline of order $k$. Verify the following.

(a) (Optional) The following recursion holds over the order,

\[ k \cdot \beta^k(x) = \left(\frac{k+1}{2} + x\right) \cdot \beta^{k-1}(x + \frac{1}{2}) + \left(\frac{k+1}{2} - x\right) \cdot \beta^{k-1}(x - \frac{1}{2}). \]

(b) The $k$-th order spline $\beta^k$ is a nonnegative, symmetric and piecewise polynomial of degree $k$, with finite support.

(c) Every B-spline has continuous spectrum of infinite support. The odd-order B-splines are non-negative in the spectral space.

(d) Demonstrate in MATLAB the curves of B-splines up to a chosen order and their Fourier transforms, and the curves obtained by DFT of the B-splines.

MATLAB: use \texttt{fftshift}.

(e) (Optional) The Gaussians are invariant under Fourier transform.

6. Use two different (iterative) methods to extract all the 4-th roots of 1.

(a) Describe the strategy for initial values and the criteria for termination associated with each method.

(b) Demonstrate the relationship between the initial values and the converged values.