CPS296.2 Geometric Optimization **March, 08 2007**

Lecture 17: Covering, Packing, and Art-Gallery Problems

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In this lecture we will continue to study covering and packing problems and briefly touch upon art-gallery problems.

17.1 Covering and Packing

We now study the problem of finding a cover for a point set by disks of unit diameter. Formally, given a set U of n points in $\mathbb{R}^2$ compute the smallest of disks of unit diameter that cover U.

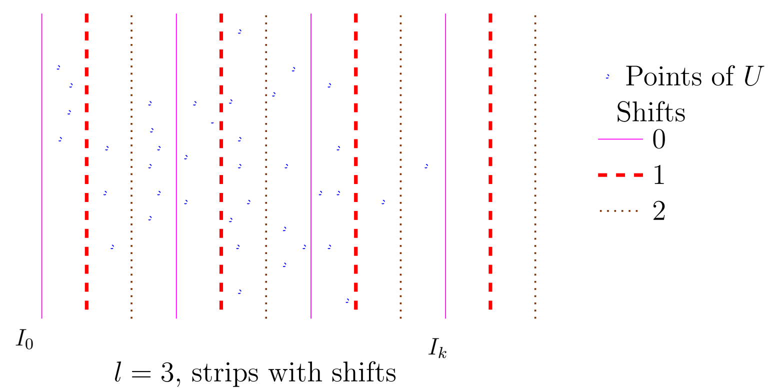

Figure 17.1: Illustration of partitioning and shifts done by the algorithm to compute minimum disk.

Let $I_0 : X = X_0$ be a vertical line such that all points of U lie to right of I_0. Let l be an integral "shifting" parameter. Add vertical lines $\{I_1, \ldots, I_k\}$ to the right I_0 so that the strips defined by lines I_{i-1}, I_i have width l, i.e. $I_j : x = x_0 + j \cdot l$. Also, let I_k be the last such line. Let P_0 denote this partition of the points of U into vertical strips. For $0 \leq i < l$, let P_i be the partition of U induced by the vertical lines $X = X_0 + jl + i$, $0 \leq j \leq k$, as shown in figure 17.1. For each P_i, let S_{ij} be the set of points that fall into the strip bounded by the lines I_j and I_{j+1} and let $S_i = \bigcup_{0 \leq j \leq k-1} S_{ij}$. Let $e_1, \ldots, e_d$ represent the standard bases for $\mathbb{R}^d$. The algorithm 1 uses the sub-division described above to compute an approximate minimum disk cover for $U \subseteq \mathbb{R}^d$. The inputs to the algorithm are the point set U, the shifting parameter l and direction parallel to which the strips are to be produced.

The algorithm MIN_DISK_COVER1 uses the algorithm MIN_DISK_COVER2 which exhaustively computes the minimum disk cover in side the hypersquare with side of length l. The following lemma establishes the approximation factor for the above algorithm.

Algorithm 1 MIN_DISK_COVER1(U, l, e_q)

Generate partitions $P_0, \ldots, P_k$ with strips parallel to the direction e_q
for $i = 1$ to $l - 1$ **do**
  for $S \in S_i$ **do**
    if $d > 1$ **then**
      D_S =MIN_DISK_COVER1(S, l, e_{q-1})
    else
      D_S =MIN_DISK_COVER2(S).
    end if
  end for
  $C_i = \bigcup_{S \in S_i} D_S$
end for
return $\min_{1 \le i \le l-1} C_i$

Lemma 1 *If the recursive calls used on strips return α-approximate disk-covers for their corresponding subsets of U, then the algorithm 1 is an $\alpha(1 + 1/l)$-approximate disk cover for U.*
Proof: Let the size of the smallest disk-cover for a point set S be $\omega(S)$. Since the recursive calls return α-approximate disk-covers,

$$|C_i| \le \alpha \sum_{s \in S_i} \omega(S). \tag{17.1}$$

Let $\omega_1, \ldots, \omega_{l-1}$ denote the number of disks in an optimal cover of U which contain points from different strips in P_i. Clearly,

$$\sum_{S \in S_i} \omega(S) \le \omega(U) + \omega_i. \tag{17.2}$$

Since a disk of unit diameter cannot be in two different ω_i's,

$$\sum_{i=0}^{l} \omega_i \le \omega(U).$$

and

$$\sum_{i=0}^{l-1} (\omega(U) + \omega_i) \le (1 + l)\omega(U). \tag{17.3}$$

So,

$$\min_i |C_i| \le \alpha \min_i \sum_{S \in S_i} \omega(S) \le \alpha \frac{1}{l} \sum_{i=0}^{l-1} \sum_{S \in S_i} \omega(S) \le \alpha(1 + \frac{1}{l})\omega(U). \tag{17.4}$$

■

In two dimensions, to cover an $l \times l$ square, $(l\sqrt{2})^2 = 2l^2$ disks are sufficient and if there are m points in this square the exhaustive search can be performed in $O((2n)^{2(l\sqrt{2})^2+1})$ time. This generalizes to higher dimensions and we have the following theorem.

Theorem 1 *For any $l \geq 1$, the given algorithm computes a $(1 + 1/l)^d$-approximate disk cover for a set of points $U \in \mathbb{R}^d$ of size n in time $O(l^d(l\sqrt{d})^d(2n)^{d(l\sqrt{d})^d+1})$.*

17.2 Art Gallery Problems

Consider an art gallery whose floor plan can be treated as a polygon P. The gallery is to be protected by guards who are modeled as points in P. Further, each guard is assumed to be able to see a portion of the polygon and all the guards together must cover the entire polygon. A natural question to ask is how many guards are necessary in order to guard the gallery ? To formalize these notions, we first define visibility. A point $v \in P$ is *visible* to a point $u \in P$, if the segment uv does not intersect the exterior of P. A set of Q points of the polygon P are said to *cover* (or *guard*) P if every point of P is visible to some point of Q. Let R be a set of points that cover P. Let $\mu(P) = \min_R |R|$ denote the minimum number of guards required to cover the polygon P. Let P_n be the set of all polygons with n vertices. Then the goal the of the art gallery problem is to find a number $g(n)$ such that:

$$g(n) = \max_{P \in P_n} \mu(P).$$

The original problem was posed by Klee in 1973, and was first solved by Chvátal in 1975. A simple proof of this problem was later presented by Fisk in 1978 based on triangulating polygons.

Theorem 2 $g(n) = \lfloor n/3 \rfloor$.
Sketch of Proof Given a polygon P with n vertices, triangulate it by adding diagonals. Create the graph G whose nodes are vertices of the polygon and edges are sides and diagonals of the triangulated polygon. Fisk showed that such a graph G is 3-colorable. Since every triangle can be guarded by a single vertex, $\lfloor n/3 \rfloor$ guards suffice. To prove $\lfloor n/3 \rfloor$ guards are necessary, a construction such as the one shown in Figure 17.2 can be used. ■

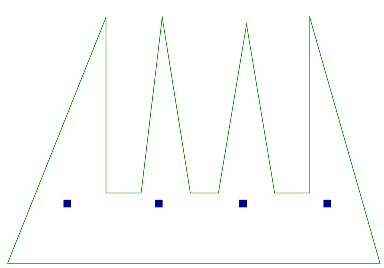

Figure 17.2: Illustration of a polygon with 12 vertices for which 4 guards are required.

There are many variations of the problem depending on location and type of guards such as:

- **Point guards:** This is the traditional problem above with guards allowed to be present anywhere in the polygon P.

- **Edge guards:** In this scheme the guards are allowed to be only on the edges of P.
- **Vertex guards:** Guards are allowed only on the vertices of P.
- **Mobile guards:** In this model the guards are allowed to patrol as opposed to being stationary in the classical model.

Although $\lfloor n/3 \rfloor$ guards suffice in general, for a given polygon the minimum number of guards required might be much less. The problem of finding the minimum number of required to cover a polygon was proved to be NP-hard [5]. They also showed that even for the variants of the problem with vertex and edge guards finding minimum number of guards is NP-hard. Furthermore, Eidenbenz [3] showed that:

Theorem 3 *There is no $(1+\epsilon)$-approximation scheme to find the minimum number of guards for the art gallery problem unless $P = NP$.*

For the vertex version of the algorithm Ghosh [4] designed an $O(\log n)$ approximation algorithm that runs in $O(n^5 \log n)$ time. For a point x belonging to a polygon P, define $V(x)$ to be the sub-polygon visible to x. Valtr [7] proved that the range space $(P, \mathcal{V})$, where $\mathcal{V} = \{V(q) : q \in P\}$, has VC-dimension between 6 and 23. Efrat and Har-Peled [2], present a $O(nc_{\text{opt}}^2 \log^4 n)$ time algorithm to find a set of vertices that guard the polygon P, whose size is within a factor $O(\log c_{\text{opt}})$, where c_{opt} is the size of the optimum number of vertices that can guard P. In their approach, they use the Bronniman and Goodrich algorithm (see lecture 16) and compute the subset of vertices that cover $\mathcal{V}$. A survey on art gallery problems can be found in [6].

Recently, there has been a surge of interest in studying variants of art gallery problems. We present a few of them below.

Visibility constraints

In this scheme, the guard's visibility is restricted to annular region so that it cannot see too near or too far away objects. Also, near the boundary walls the visibility is restricted to an angular range as shown in Figure 17.3.

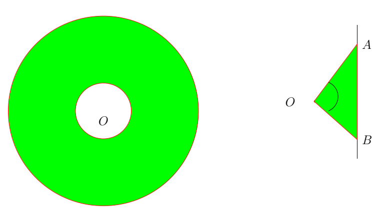

Figure 17.3: Art gallery problem with modified visibility constraints.

Robot localization

Let B be a robot, which is placed in an polygon P. The robot has a map of P, but does not know the exact location q_0 in P that it is placed at. From it's position the robot can only see the visibility polygon

$V(q_0) \subseteq P$. In general, just knowing $V(q_0)$ will not be sufficient to determine q, as there may have multiple points whose visibility polygons might be congruent. So the robot is allowed to move by a suitable amount and sense the environment to determine it's position in P. However, it is desirable to perform localization with minimum amount of motion. Let $H = \{q_i : V(q_i)$ is congruent to $V(q_0)\}$, and let d be the minimum amount of motion required by the robot to verify it's initial position $q_0$by sensing. Dudek et. al [1] show that robot localization with minimum travel is NP-hard and present a polynomial time approximation algorithm with a travel distance of at most $(|H| - 1)d$.

References

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