Section: Properties of Context-free Languages

Which of the following languages are CFL?

- \( L = \{ a^n b^n c^j \mid 0 < n \leq j \} \)
- \( L = \{ a^n b^j a^n b^j \mid n > 0, j > 0 \} \)
- \( L = \{ a^n b^j a^k b^p \mid n + j \leq k + p, n > 0, j > 0, k > 0, p > 0 \} \)
Pumping Lemma for Regular Language’s: Let $L$ be a regular language, Then there is a constant $m$ such that $w \in L$, $|w| \geq m$, $w = xyz$ such that

- $|xy| \leq m$
- $|y| \geq 1$
- for all $i \geq 0$, $xy^iz \in L$
Pumping Lemma for CFL’s Let $L$ be any infinite CFL. Then there is a constant $m$ depending only on $L$, such that for every string $w$ in $L$, with $|w| \geq m$, we may partition $w = uvxyz$ such that:

- $|vxy| \leq m$, (limit on size of substring)
- $|vy| \geq 1$, ($v$ and $y$ not both empty)

For all $i \geq 0$, $uv^i xy^i z \in L$

- **Proof: (sketch)** There is a CFG $G$ s.t. $L = L(G)$.
  Consider the parse tree of a long string in $L$.
  For any long string, some nonterminal $N$ must appear twice in the path.
Example: Consider

$L = \{a^nb^nc^n : n \geq 1\}$. Show $L$ is not a CFL.

- Proof: (by contradiction)
  Assume $L$ is a CFL and apply the pumping lemma.
  Let $m$ be the constant in the pumping lemma and consider
  $w = a^mb^mc^m$. Note $|w| \geq m$.
  Show there is no division of $w$ into $uvxyz$ such that $|vy| \geq 1$, $|vxy| \leq m$,
and $uv^ixy^iz \in L$ for $i = 0, 1, 2, \ldots$. 
Thus, there is no breakdown of $w$ into $uvxyz$ such that $|vy| \geq 1$, $|vxy| \leq m$ and for all $i \geq 0$, $uv^ixy^iz$ is in $L$. Contradiction, thus, $L$ is not a CFL. Q.E.D.
Example Why would we want to recognize a language of the type \( \{a^n b^n c^n : n \geq 1\} \)?

Example: Consider 
\( L = \{a^n b^n c^p : p > n > 0\} \). Show \( L \) is not a CFL.

- Proof: Assume \( L \) is a CFL and apply the pumping lemma. Let \( m \) be the constant in the pumping lemma and consider 
\( w = \) ____________ Note \( |w| \geq m \).
Show there is no division of \( w \) into \( uvxyz \) such that \( |vy| \geq 1, |vxy| \leq m \), and \( uv^i xy^i z \in L \) for \( i = 0, 1, 2, \ldots \).
Example: Consider $L = \{a^j b^k : k = j^2\}$. Show $L$ is not a CFL.

- Proof: Assume $L$ is a CFL and apply the pumping lemma. Let $m$ be the constant in the pumping lemma and consider

  $w = \_\_\_\_

  Show there is no division of $w$ into $uvxyz$ such that $|vy| \geq 1$, $|vxy| \leq m$, and $uv^i xy^i z \in L$ for $i = 0, 1, 2, \ldots$.

  Case 1: Neither $v$ nor $y$ can contain 2 or more distinct symbols. If $v$ contains $a$’s and $b$’s, then $uv^2 xy^2 z \notin L$ since there will be $b$’s before $a$’s. Thus, $v$ and $y$ can be only $a$’s, and $b$’s (not mixed).
Example: Consider
$L = \{w\bar{w}w : w \in \Sigma^*\}$, $\Sigma = \{a, b\}$, where $\bar{w}$ is the string $w$ with each occurrence of $a$ replaced by $b$ and each occurrence of $b$ replaced by $a$. Show $L$ is not a CFL.

• Proof: Assume $L$ is a CFL and apply the pumping lemma. Let $m$ be the constant in the pumping lemma and consider
$w = __________$

Show there is no division of $w$ into $uvxyz$ such that $|vy| \geq 1$, $|vxy| \leq m$, and $uv^ixy^iz \in L$ for $i = 0, 1, 2, \ldots$. 
Example: Consider $L = \{a^n b^p b^p a^n\}$. $L$ is a CFL. The pumping lemma should apply!

Let $m \geq 4$ be the constant in the pumping lemma. Consider $w = a^m b^m b^m a^m$.

We can break $w$ into $uvxyz$, with:
Theorem CFL’s are closed under union, concatenation, and star-closure.

Proof:

Given 2 CFG $G_1 = (V_1, T_1, S_1, P_1)$ and $G_2 = (V_2, T_2, S_2, P_2)$

- Union:

  Construct $G_3$ s.t. $L(G_3) = L(G_1) \cup L(G_2)$.

  $G_3 = (V_3, T_3, S_3, P_3)$
– Concatenation:

Construct $G_3$ s.t. $L(G_3) = L(G_1) \circ L(G_2)$.

$G_3 = (V_3, T_3, S_3, P_3)$

– Star-Closure

Construct $G_3$ s.t. $L(G_3) = L(G_1)^*$

$G_3 = (V_3, T_3, S_3, P_3)$
Theorem CFL’s are NOT closed under intersection and complementation.

• Proof:
  – Intersection:
– Complementation:
Theorem: CFL’s are closed under regular intersection. If $L_1$ is CFL and $L_2$ is regular, then $L_1 \cap L_2$ is CFL.

• Proof: (sketch) We take a NPDA for $L_1$ and a DFA for $L_2$ and construct a NPDA for $L_1 \cap L_2$.

$M_1 = (Q_1, \Sigma, \Gamma, \delta_1, q_0, z, F_1)$ is an NPDA such that $L(M_1) = L_1$.

$M_2 = (Q_2, \Sigma, \delta_2, q_0, F_2)$ is a DFA such that $L(M_2) = L_2$.

Example of replacing arcs (NOT a Proof!):
We must formally define $\delta_3$. If

then

Must show

if and only if
Questions about CFL:

1. Decide if CFL is empty?

2. Decide if CFL is infinite?
Example: Consider 
\[ L = \{a^{2n}b^{2m}c^nd^m : n, m \geq 0\} \]. Show \( L \) is not a CFL.

- Proof: Assume \( L \) is a CFL and apply the pumping lemma. Let \( m \) be the constant in the pumping lemma and consider 
\[ w = a^{2m}b^{2m}c^md^m. \]

Show there is no division of \( w \) into \( uvxyz \) such that \( |vy| \geq 1 \), \( |vxy| \leq m \), and \( uv^i xy^i z \in L \) for \( i = 0, 1, 2, \ldots \).

Case 1: Neither \( v \) nor \( y \) can contain 2 or more distinct symbols. If \( v \) contains \( a \)'s and \( b \)'s, then \( uv^2xy^2z \notin L \) since there will be \( b \)'s before \( a \)'s.

Thus, \( v \) and \( y \) can be only \( a \)'s, \( b \)'s, \( c \)'s, or \( d \)'s (not mixed).

Case 2: \( v = a^{t_1} \), then \( y = a^{t_2} \) or \( b^{t_3} \) \((|vxy| \leq m)\)

If \( y = a^{t_2} \), then
\[ uv^2xy^2z = a^{2m+t_1+t_2}b^{2m}c^md^m \notin L \text{ since } t_1 + t_2 > 0, \text{ the number of } a\text{'s is not twice the number of } c\text{'s.} \]

If \( y = b^{t_3} \), then
\[ uv^2xy^2z = a^{2m+t_1}b^{2m+t_3}c^md^m \notin L \text{ since } t_1 + t_3 > 0, \text{ either the number of } a\text{'s (denoted } n(a)\text{) is not twice } n(c) \text{ or } n(b) \text{ is not twice } n(d). \]

Case 3: \( v = b^{t_1} \), then \( y = b^{t_2} \) or \( c^{t_3} \)

If \( y = b^{t_2} \), then
\[ uv^2xy^2z = a^{2m}b^{2m+t_1+t_2}c^md^m \notin L \text{ since } t_1 + t_2 > 0, n(b) > 2*n(d). \]

If \( y = c^{t_3} \), then
\[ uv^2xy^2z = a^{2m}b^{2m+t_1}c^{m+t_3}d^m \notin L \text{ since } t_1 + t_3 > 0, \text{ either } n(b) > 2*n(d) \text{ or } 2*n(c) > n(a). \]

Case 4: \( v = c^{t_1} \), then \( y = c^{t_2} \) or \( d^{t_3} \)

If \( y = c^{t_2} \), then
\[ uv^2xy^2z = a^{2m}b^{2m}c^{m+t_1+t_2}d^m \notin L \text{ since } t_1 + t_2 > 0, 2*n(c) > n(a). \]

If \( y = d^{t_3} \), then
\(uv^2xy^2z = a^{2m}b^2mc^{m+t_1}d^{m+t_3} \notin L\) since 
\(t_1 + t_3 > 0\), either \(2*n(c) > n(a)\) or 
\(2*n(d) > n(b)\).

**Case 5:** \(v = d^{t_1}\), then \(y = d^{t_2}\)
then \(uv^2xy^2z = a^{2m}b^2mc^{m}d^{m+t_1+t_2} \notin L\)
since \(t_1 + t_2 > 0\), \(2*n(d) > n(c)\).

Thus, there is no breakdown of \(w\) 
into \(uvxyz\) such that \(|vy| \geq 1\), 
\(|vxy| \leq m\) and for all \(i \geq 0\), \(uv^i xy^i z\) is 
in \(L\). Contradiction, thus, \(L\) is not a 
CFL. Q.E.D.