Section: Decidability

Computability A function $f$ with domain $D$ is \textit{computable} if there exists some TM $M$ such that $M$ computes $f$ for all values in its domain.

Decidability A problem is \textit{decidable} if there exists a TM that can answer yes or no to every statement in the domain of the problem.
The Halting Problem

Domain: set of all TMs and all strings \( w \).

Question: Given coding of M and w, does M halt on w?
Theorem The halting problem is undecidable.

Proof: (by contradiction)

• Assume there is a TM H (or algorithm) that solves this problem.
  TM H has 2 final states, $q_y$ represents yes and $q_n$ represents no.

  $$H(w_M, w) = \begin{cases} 
  \text{halts } q_y \text{ if } M \text{ halts on } w \\
  \text{halts } q_n \text{ if } M \text{ doesn’t halt on } w 
  \end{cases}$$

  TM H always halts in a final state.
Construct TM $H'$ from $H$

$$H'(w_M, w) = \begin{cases} 
\text{halts} & \text{if } M \text{ not halt on } w \\
\text{not halt} & \text{if } M \text{ halts on } w 
\end{cases}$$

Construct TM $\hat{H}$ from $H'$

$$\hat{H}(w_M) = \begin{cases} 
\text{halts} & \text{if } M \text{ not halt on } w_M \\
\text{not halt} & \text{if } M \text{ halts on } w_M 
\end{cases}$$

Note that $\hat{H}$ is a TM.

There is some encoding of it, say $\hat{w}_{\hat{H}}$.

What happens if we run $\hat{H}$ with input $\hat{w}_{\hat{H}}$?
Theorem If the halting problem were decidable, then every recursively enumerable language would be recursive. Thus, the halting problem is undecidable.

Proof: Let $L$ be an RE language over $\Sigma$.
Let $M$ be the TM such that $L=L(M)$.
Let $H$ be the TM that solves the halting problem.
A problem A is reduced to problem B if the decidability of B follows from the decidability of A. Then if we know B is undecidable, then A must be undecidable.
State-entry problem Given TM $M=\langle Q, \Sigma, \Gamma, \delta, q_0, B, F \rangle$, state $q \in Q$, and string $w \in \Sigma^*$, is state $q$ ever entered when $M$ is applied to $w$?

This is an undecidable problem!

- **Proof:**

  TM $E$ solves state-entry problem

  $$E'(w_M, w) = \begin{cases} 
  M \text{ halts on } w & \text{if } \? \\
  M \text{ doesn't halt on } w & \text{if } \? 
  \end{cases}$$