Lecture notes ?: Solving (mixed) integer programs using branch and bound

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We now turn to solving (mixed) integer programs. There are several different approaches to this; most of them are based on solving various LP relaxations of the integer program, since solving linear programs is easier than solving integer programs. In fact, solving integer programs is NP-hard (we have already seen how to model numerous NP-hard problems as integer programs), whereas linear programs can be solved in polynomial time.

Arguably the simplest approach to solving (mixed) integer programs is to use *branch-and-bound*. This approach works as follows. We solve the LP relaxation of the (mixed) integer program. If we obtain an optimal solution in which all the variables (or, in the case of a mixed integer program, all the variables that are required to take integer values) are set to integer values, then we are done. Otherwise, take some variable x_i that is required to take an integer value, but is set to a noninteger value r in the solution to the LP relaxation. We know that either $x \leq \lfloor r \rfloor$, or $x \geq \lceil r \rceil$. We then create two new problems in which one of these constraints is added, and continue.

It helps to see an example. Let us consider again the modified painting example:

maximize $3x_1 + 2x_2$
subject to
$4x_1 + 2x_2 \leq 15$
$x_1 + 2x_2 \leq 8$
$x_1 + x_2 \leq 5$
$x_1 \geq 0$, integer; $x_2 \geq 0$, integer
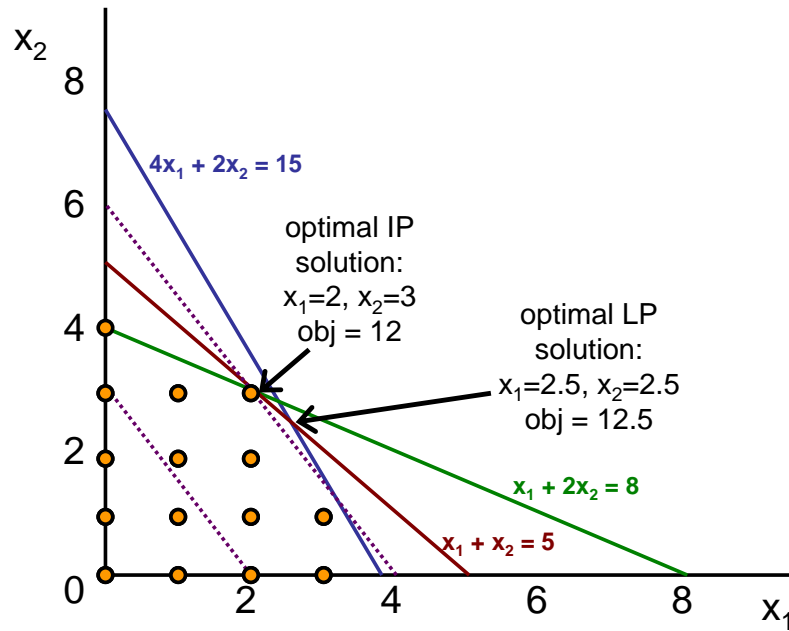
Figure 1 illustrates this integer program.

Figure 1: Graphical representation of the modified painting problem instance with integrality constraints.

The optimal solution to this integer program is to set $x_1 = 2, x_2 = 3$ (with an objective value of 12). However, the optimal solution to the LP relaxation is to set $x_1 = 2.5, x_2 = 2.5$ (with an objective value of 12.5). Because both variables are required to take integer values but currently have fractional values, branch-and-bound can branch on either. Let us say that we branch on x_1 first. We create two new integer programs:

maximize $3x_1 + 2x_2$
subject to
$4x_1 + 2x_2 \leq 15$
$x_1 + 2x_2 \leq 8$
$x_1 + x_2 \leq 5$
$x_1 \geq 3$
$x_1 \geq 0$, integer; $x_2 \geq 0$, integer

as well as

maximize $3x_1 + 2x_2$
subject to
$4x_1 + 2x_2 \leq 15$
$x_1 + 2x_2 \leq 8$
$x_1 + x_2 \leq 5$
$x_1 \leq 2$
$x_1 \geq 0$, integer; $x_2 \geq 0$, integer

We now have a choice as to which of these two we solve first. Let us say that we continue on the first one first. Again, we solve the LP relaxation. The solution is $x_1 = 3, x_2 = 1.5$, with an objective value of 12. This is still not a feasible solution to the integer program, because x_2 takes a fractional value. So we can branch on x_2 next, obtaining the following two integer programs:

maximize $3x_1 + 2x_2$
subject to
$4x_1 + 2x_2 \leq 15$
$x_1 + 2x_2 \leq 8$
$x_1 + x_2 \leq 5$
$x_1 \geq 3$
$x_2 \geq 2$
$x_1 \geq 0$, integer; $x_2 \geq 0$, integer

and

maximize $3x_1 + 2x_2$
subject to
$4x_1 + 2x_2 \leq 15$
$x_1 + 2x_2 \leq 8$
$x_1 + x_2 \leq 5$
$x_1 \geq 3$
$x_2 \leq 1$
$x_1 \geq 0$, integer; $x_2 \geq 0$, integer

Also, we should not forget about the integer program

maximize $3x_1 + 2x_2$
subject to
$4x_1 + 2x_2 \leq 15$
$x_1 + 2x_2 \leq 8$
$x_1 + x_2 \leq 5$
$x_1 \leq 2$
$x_1 \geq 0$, integer; $x_2 \geq 0$, integer

that we have yet to explore. Indeed, we could continue on any one of these three integer programs. Continuing on the one with $x_1 \leq 2$ would correspond to doing *breadth-first* search, whereas continuing on either of the other two would correspond to doing *depth-first* search. Let us go with depth first, continuing on the integer program with $x_2 \geq 2$. This program is infeasible (it is not possible to satisfy the first constraint as well as the added constraints). So we can forget about this branch. Let us consider the integer program with $x_2 \leq 1$ next. Solving the LP relaxation of this, we obtain $x_1 = 3.25, x_2 = 1$, with an objective value of 11.75. x_1 has become fractional again! Branching on x_1 leads to the programs:

maximize $3x_1 + 2x_2$
subject to
$4x_1 + 2x_2 \leq 15$
$x_1 + 2x_2 \leq 8$
$x_1 + x_2 \leq 5$
$x_1 \geq 3$
$x_2 \leq 1$
$x_1 \geq 4$
$x_1 \geq 0$, integer; $x_2 \geq 0$, integer

and

maximize $3x_1 + 2x_2$
subject to
$4x_1 + 2x_2 \leq 15$
$x_1 + 2x_2 \leq 8$
$x_1 + x_2 \leq 5$
$x_1 \geq 3$
$x_2 \leq 1$
$x_1 \leq 3$
$x_1 \geq 0$, integer; $x_2 \geq 0$, integer

We continue our depth-first search on these two integer programs. The LP relaxation of the first one is infeasible. The LP relaxation of the second one leads to the solution $x_1 = 3, x_2 = 1$, with an objective value of 11. Because the variables are set to integer values, we are done here. All that is left to explore is the integer program

maximize $3x_1 + 2x_2$
subject to
$4x_1 + 2x_2 \leq 15$
$x_1 + 2x_2 \leq 8$
$x_1 + x_2 \leq 5$
$x_1 \leq 2$
$x_1 \geq 0$, integer; $x_2 \geq 0$, integer

The optimal solution to the LP relaxation is $x_1 = 2, x_2 = 3$, with an objective value of 12. Because the variables are set to integers, we do not need to branch further. Also, this integer feasible solution is better than the one that we found before (which gave us 11). There is nothing left to explore, so we know that $x_1 = 2, x_2 = 3$ is the optimal solution to the integer program.

Let us consider what would have happened if we had explored the possibilities in a different order. Consider again the situation that we were in after the first branch: we had the integer programs

maximize $3x_1 + 2x_2$
subject to
$4x_1 + 2x_2 \leq 15$
$x_1 + 2x_2 \leq 8$
$x_1 + x_2 \leq 5$
$x_1 \geq 3$
$x_1 \geq 0$, integer; $x_2 \geq 0$, integer

and

maximize $3x_1 + 2x_2$
subject to
$4x_1 + 2x_2 \leq 15$
$x_1 + 2x_2 \leq 8$
$x_1 + x_2 \leq 5$
$x_1 \leq 2$
$x_1 \geq 0$, integer; $x_2 \geq 0$, integer

As it turns out, we chose an unfortunate order of exploring the options here. If we had chosen to explore the second program first, we would have found the integer feasible solution $x_1 = 2, x_2 = 3$, with an objective value of 12, right away. Then, when we solve the LP relaxation of the first program next, we find that the optimal solution $x_1 = 3, x_2 = 1.5$ has an objective value of only 12. Because this time, we have already found an integer feasible solution with objective value 12, we realize immediately that there is no point in branching further: as we add more constraints, certainly the objective cannot rise above 12, so we will not find anything better. So we can immediately conclude that we are done. This is the "bound" in branch-and-bound: we keep track of the best integer feasible solution that we have found so far, and we do not need to explore any programs whose LP-relaxation value is less than (or equal to) the value of our current best solution—because the LP relaxation is an upper bound on any solution value that we will find by branching further on such a program.