

12 Recursion

We now describe how recurrence relations arise from recursive algorithms, and begin to look at ways of solving them. We have just learned one method that can sometimes be used to solve such a relation, namely Mathematical Induction. In fact, we can think of recursion as backwards induction.

The towers of Hanoi. Recurrence relations naturally arise in the analysis of the towers of Hanoi problems. Here we have three pegs, A , B , C , and initially n disks at A , sorted from large to small; see Figure 10. The task is to move the n disks from A to C , one by one, without ever placing a larger disk onto a smaller disk. The following

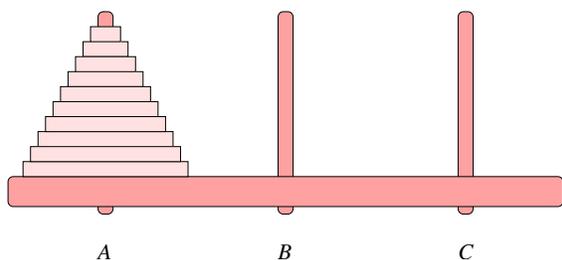


Figure 10: We have a sorted stack of disks at A and use B for temporary storage to move one disk at a time to C . We need B to avoid any inversions among the disks.

three steps solve this problem:

- recursively move $n - 1$ disks from A to B ;
- move the n -th disk from A to C ;
- recursively move $n - 1$ disks from B to C .

When we move disks from one peg to another, we use the third peg to help. For the main task, we use B to help. For the first step, we exchange the roles of B and C , and for the third step, we exchange the roles of A and B . The number of moves is given by the solution to the recurrence relation

$$M(n) = 2M(n - 1) + 1,$$

with initial condition $M(0) = 0$. We may use induction to show that $M(n) = 2^n - 1$.

Loan payments. Another example in which recurrence relations naturally arise is the repayment of loans. This

is an iterative process in which we alternate the payment of a constant sum with the accumulation of interest. The iteration ends when the entire loan is payed off. Suppose A_0 is the initial amount of your loan, m is your monthly payment, and p is the annual interest payment rate. What is the amount you owe after n months? We can express it in terms of the amount owed after $n - 1$ months:

$$T(n) = \left(1 + \frac{p}{12}\right)T(n - 1) - m.$$

This is a recurrence relation, and figuring out how much you owe is the same as solving the recurrence relation. The number that we are most interested in is n_0m , where n_0 is the number of months it takes to get $T(n_0) = 0$. Instead of attacking this question directly, let us look at a more abstract, mathematical setting.

Iterating the recursion. Consider the following recurrence relation,

$$T(n) = rT(n - 1) + a,$$

where r and a are some fixed real numbers. For example, we could set $r = 1 + \frac{p}{12}$ and $a = -m$ to get the recurrence that describes how much money you owe. After replacing $T(n)$ by $rT(n - 1) + a$, we may take another step and replace $T(n - 1)$ by $rT(n - 2) + a$ to get $T(n) = r(rT(n - 2) + a) + a$. Iterating like this, we get

$$\begin{aligned} T(n) &= rT(n - 1) + a \\ &= r^2T(n - 2) + ra + a \\ &= r^3T(n - 3) + r^2a + ra + a \\ &\vdots \\ &= r^nT(0) + a \sum_{i=0}^{n-1} r^i. \end{aligned}$$

The first term on the right hand side is easy, namely r^n times the initial condition, say $T(0) = b$. The second term is a sum, which we now turn into a nicer form.

Geometric series. The sequence of terms inside a sum of the form $\sum_{i=0}^{n-1} r^i$ is referred to as a *geometric series*. If $r = 1$ then this sum is equal to n . To find a similarly easy expression for other values of r , we expand both the sum and its r -fold multiple:

$$\begin{aligned} \sum_{i=0}^{n-1} r^i &= r^0 + r^1 + r^2 + \dots + r^{n-1}; \\ r \sum_{i=0}^{n-1} r^i &= r^1 + r^2 + \dots + r^{n-1} + r^n. \end{aligned}$$

Subtracting the second line from the first, we get

$$(1-r) \sum_{i=0}^{n-1} r^i = r^0 - r^n$$

and therefore $\sum_{i=0}^{n-1} r^i = \frac{1-r^n}{1-r}$. Now, this allows us to rewrite the solution to the recurrence as

$$T(n) = r^n b + a \frac{1-r^n}{1-r},$$

where $b = T(0)$ and $r \neq 1$. Let us consider the possible scenarios:

Case 1. $r = 0$. Then, $T(n) = a$ for all n .

Case 2. $0 < r < 1$. Then, $\lim_{n \rightarrow \infty} r^n = 0$. Therefore, $\lim_{n \rightarrow \infty} T(n) = \frac{a}{1-r}$.

Case 3. $r > 1$. The factors r^n of b and $\frac{r^n-1}{r-1}$ of a both grow with growing n . For positive values of a and b , we can expect $T(n) = 0$ for a negative value of n . Multiplying with $r-1$, we get $r^n b(r-1) + ar^n - a = 0$ or, equivalently, $r^n (br - b + a) = a$. Dividing by $br - b + a$, we get $r^n = \frac{a}{br-b+a}$, and taking the logarithm to the base r , we get

$$n = \log_r \left(\frac{a}{br - b + a} \right).$$

For positive values of a and b , we take the logarithm of a positive number smaller than one. The solution is a negative number n .

We note that the loan example falls into Case 3, with $r = 1 + \frac{p}{12} > 1$, $b = A_0$, and $a = -m$. Hence, we are now in a position to find out after how many months it takes to pay back the loan, namely

$$n_0 = \log_r \left(\frac{m}{m - A_0 \frac{p}{12}} \right).$$

This number is well defined as long as $m > A_0 \frac{p}{12}$, which means your monthly payment should exceed the monthly interest payment. It better happen, else the amount you owe grows and the day in which the loan will be payed off will never arrive.

First-order linear recurrences. The above is an example of a more general class of recurrence relations, namely the *first-order linear recurrences* that are of the form

$$T(n) = f(n)T(n-1) + g(n).$$

For the constant function $f(n) = r$, we have

$$\begin{aligned} T(n) &= r^n T(0) + \sum_{i=0}^{n-1} r^i g(n-i) \\ &= r^n T(0) + \sum_{i=0}^{n-1} r^{n-i} g(i). \end{aligned}$$

We see that if $g(n) = a$, then we have the recurrence we used above. We consider the example $T(n) = 2T(n-1) + n$ in which $r = 2$ and $g(i) = i$. Hence,

$$\begin{aligned} T(n) &= 2^n T(0) + \sum_{i=0}^{n-1} \frac{i}{2^{n-i}} \\ &= 2^n T(0) + \frac{1}{2^n} \sum_{i=0}^{n-1} i 2^i. \end{aligned}$$

It is not difficult to find a closed form expression for the sum. Indeed, it is the special case for $x = 2$ of the following result.

CLAIM. For $x \neq 1$, we have

$$\sum_{i=1}^n i x^i = \frac{nx^{n+2} - (n-1)x^{n+1} + x}{(1-x)^2}.$$

PROOF. One way to prove the relation is by induction. Writing $R(n)$ for the right hand side of the relation, we have $R(1) = x$, which shows that the claimed relation holds for $n = 1$. To make the step from $n-1$ to n , we need to show that $R(n-1) + x^n = R(n)$. It takes but a few algebraic manipulations to show that this is indeed the case. \square

Summary. Today, we introduced recurrence relations. To find the solution, we often have to define $T(n)$ in terms of $T(n_0)$ rather than $T(n-1)$. We also saw that different recurrences can have the same general form. Knowing this will help us to solve new recurrences that are similar to others that we have already seen.