Questions may continue on the back. Please write clearly. What I cannot read, I will not grade. Typed homework is preferable. A good compromise is to type the words and write the math by hand.

1. The Matlab file population.m available on the class homework web page contains the definition of the function population we developed in class to describe a population that changes stochastically. You will also need the files binomialSample.m and poissonSample.m.

   (a) Assuming that the birth rate is 0 and the death rate is 0.01, write code to estimate the probability distribution of the number of years (that is, iterations of the dynamic system) that it takes for the population to become extinct when starting with 100 individuals.

   To this end, run population 1000 times (this will take a while, because of the inefficient memory management used in the function). For each run, store into a vector len (preallocate!) the number of iterations that were necessary for extinction (how can you infer this number from the output y from population?), and then draw a histogram of the values in len with the command

   \[ \text{hist(len)} \]

   Let the first instruction of your code be

   \[ \text{rand(‘seed’, 0)} \]

   so that the pseudo-random number generator is reset, and everyone’s plots are ideally equal to each other. Hand in your code and your histogram.

   (b) What are the sample mean and standard deviation of the entries in len? [Hint: doc mean and doc std.]

   (c) Can you justify, approximately, the value you obtain for the mean? [Hint: what is the average behavior of this stochastic system?]

2. Consider the experiment of tossing a single coin once. Let \( p \) be the probability of the event \( E \) that the coin lands on head. Let \( F \) denote the event that the coin lands on tail. Find all the values of \( p \) for which the two events \( E \) and \( F \) are independent. Explain your answer.

3. [Make sure you understand the distinction between universe and event space before you answer this question.]

   Consider the experiment of tossing a coin twice. Let \( HT \) denote the outcome in which the first coin toss lands on head, and the second lands on tail. Denote other outcomes with similar notation.

   (a) Specify the universe and event space for this experiment, and state the number of elements in the event space.

   (b) Two events are disjoint when their intersection is empty. Are the singleton events \( \{ HT \} \) and \( \{ TT \} \) in the experiment above disjoint?

   (c) Assume that the coin is fair. That is, all outcomes have equal probability. Determine whether the two events \( E = \{ TH, TT \} \) and \( F = \{ HT, TH \} \) are dependent or independent. Do so in three different ways, involving respectively \( P(E \cap F) \), \( P(E \mid F) \), and \( P(F \mid E) \).

   (d) Ditto for \( E = \{ TH, TT \} \) and \( F = \{ HH, HT, TH \} \).
4. Let $P(F) > 0$. What can you say about $P(E)$ if $P(E \mid F) = P(F)$ and $E$ and $F$ are independent?

5. Two random variables $X$, with domain $\mathcal{X} = \{0, 1\}$, and $Y$, with domain $\mathcal{Y} = \{1, 2, 3\}$, have the joint probability distribution $p_{X,Y}(x, y)$ specified by the following table:

<table>
<thead>
<tr>
<th></th>
<th>$Y = 1$</th>
<th>$Y = 2$</th>
<th>$Y = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X = 0$</td>
<td>0.2</td>
<td>0.08</td>
<td>0.12</td>
</tr>
<tr>
<td>$X = 1$</td>
<td>0.36</td>
<td>0.06</td>
<td>0.18</td>
</tr>
</tbody>
</table>

Do all your calculations to four significant figures.

(a) Write the marginal distributions $p_X(x)$ and $p_Y(y)$.

(b) Show that the variables $X$ and $Y$ above are mutually dependent.

(c) Write a $2 \times 3$ table with the conditional probabilities $p_{Y \mid X}(y \mid x)$.

(d) What is the value of $\sum_{y \in \mathcal{Y}} p_{Y \mid X}(y \mid x)$ for any conditional probability $p_{Y \mid X}$ between two random variables?

(e) Explain your previous answer intuitively.

(f) Write a $2 \times 3$ table with the conditional probabilities $p_{X \mid Y}(x \mid y)$.

(g) Would knowing the outcome $Y = 2$ change your decision on whether to bet that $X = 0$ or that $X = 1$? Why or why not?

6. The questions in this problem are sample midterm questions. They are meant to give you the flavor of the questions on the exam, not of its length or scope. Your answers for this problem will not be graded, and you are not required to hand them in. No sample answers are provided. These questions are given merely to allow you to practice for the exam. The midterm will be on Linear, Deterministic, Stationary, Discrete Dynamic Systems (LDSDDS), stochastic dynamic systems, and discrete probabilities, through page 66 (the subsection on Moments is not on the midterm).

(a) Compute the matrix product $A^T A$ where

$$A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}.$$ 

(b) What is the inner product of the following two vectors?

$$x = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$ 

(c) When is a linear, discrete, dynamic system stationary?

(d) Give the first three values $y(0), \ldots, y(2)$ of the free response of the system

$$x(n+1) = -\frac{1}{2}x(n) - 1$$
$$y(n) = 2x(n)$$

with $x(0) = 2$. 

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(e) Can a recursion of the following form have solution $2, 1.5, 0.75, 0.5, 0.75, 1, 1.25, \ldots$?

$$
\begin{align*}
y(0) &= y_0 \\
y(n + 1) &= ay(n) + b
\end{align*}
$$

Either give the values $y_0$, $a$, $b$ that achieve this solution, or explain why a recursion of this form cannot have the solution above.

(f) What is a stable equilibrium?

(g) What is the expression for the Bernoulli distribution on three independent trials on the universe $\Omega = \{H, T\}$ and with $P(H) = p$ and $P(T) = q$?

(h) What is the expression for the binomial distribution on the same three trials?

(i) Define the conditional probability of event $E$ given event $F$.

(j) The joint probability distribution $p_{X,Y}(x, y)$ of two random variables $X$ and $Y$, both with domain $\{0, 1\}$, is as follows:

<table>
<thead>
<tr>
<th></th>
<th>$Y = 1$</th>
<th>$Y = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X = 0$</td>
<td>0.2</td>
<td>0.3</td>
</tr>
<tr>
<td>$X = 1$</td>
<td>0.4</td>
<td>0.1</td>
</tr>
</tbody>
</table>

Write the marginal distributions of these two random variables.

(k) Are the two random variables in the previous question independent? Why or why not?