2 Introductory Examples

Populations of animals, plants, cells, or other biological entities grow and shrink as a result of births and deaths. A simple way to describe these processes is by a *Deterministic, Discrete, Dynamic System* (DDDS), that is, an equation that postulates that the population is observed at a sequence of discrete time instants $n = 0, 1, \ldots$. The number of individuals in the population at time n is denoted by, say, $a(n)$, and some mathematical function f relates $a(n)$ to its value in the next time instant $n+1$:

$$a(n+1) = f(a(n), n) \ ,$$

where the second argument, n, to the function f expresses the fact that the function that relates $a(n)$ to $a(n+1)$ may be different for different times n.

Populations that change by more complex mechanisms may make $a(n+1)$ depend on more past values of a, for example:

$$a(n+1) = f(a(n), a(n-1), n) \ .$$

In addition, there may be different populations that vie for the same environmental resources (such as food or light), or that may prey upon each other. In that case, the size of one population at time $n+1$ may depend on the sizes of both at one or more previous points in time. For instance:

$$\begin{aligned} a_1(n+1) &= f_1(a_1(n), a_2(n-1), n) \\ a_2(n+1) &= f_2(a_1(n), a_2(n-1), n) \ . \end{aligned}$$

These systems are *dynamic* in that populations change over time. They are *discrete* because time is thought of as changing in quanta (usually, but not necessarily, equal to each other). They are *deterministic* in the sense that if we know past population sizes then we assume that we can predict the current values exactly. We will remove the assumptions of discrete and deterministic behavior later on. For now, we consider a few of the simplest examples of DDDS to motivate the concepts that follow.

2.1 DDDS Example 1: The Sand-Hill Crane

The Florida Sand-Hill Crane is a bird that reproduces and dies at rates that depend on environmental conditions, the existence of predators, and other factors. These include the raising of chicks in captivity, and their subsequent release into the environment for conservation purposes, a procedure called *hacking*.

In a simple (and somewhat simplistic) model, the number of birds that are born is presumably proportional to population, and so is the number of birds that die. If there are $a(n)$ birds in year n and $u(n)$ chicks are released that year, then

$$a(n+1) = a(n) + ba(n) - da(n) + u(n)$$

or, more compactly,

$$a(n+1) = (1+r)a(n) + u(n) \tag{1}$$

where the *growth rate*

$$r = b - d$$

accounts for the difference between birth rate b and death rate d and is assumed, in this simple model, to be the same at all times. Depending on the environment, the birth rate b can be greater than, equal to, or smaller than the death rate d, so the growth rate r can be positive, zero, or negative. Values for r have been reported to lie in the following interval:[7]

$$-0.0382 \leq r \leq 0.0194 \ .$$

Suppose we start with $a(0)$ birds at the beginning of a multi-year observation interval, and let

$$R = 1 + r$$

for simplicity. Then the following years the population, in the absence of hacking (that is, when $u(n) = 0$), is

$$\begin{aligned} a(1) &= Ra(0) \\ a(2) &= Ra(1) = R(Ra(0)) = R^2 a(0) \\ a(3) &= Ra(2) = R(R^2 a(0)) = R^3 a(0) \end{aligned}$$

and the pattern is clear:

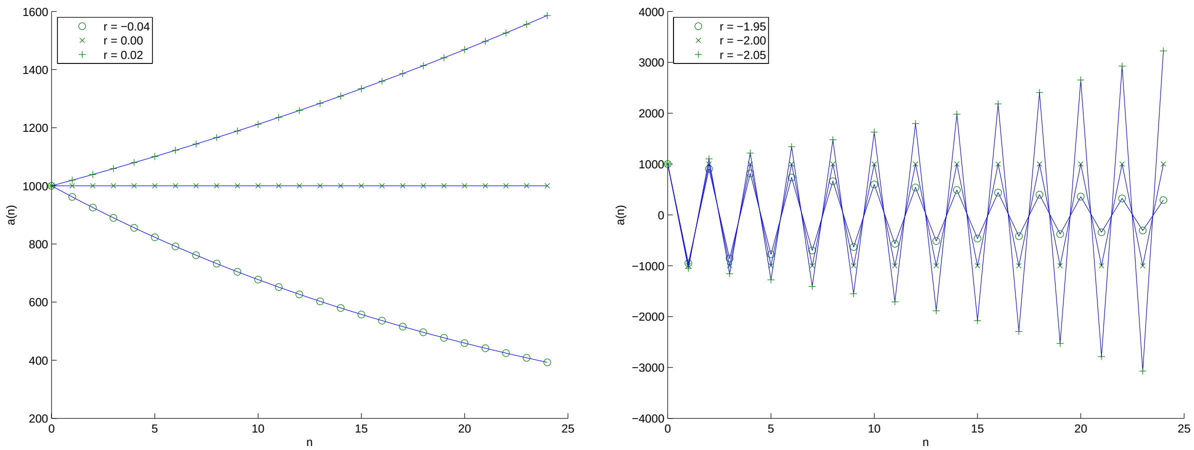
$$a(n) = R^n a(0) \ . \tag{2}$$

We have *solved* the dynamic system in the case $u(n) = 0$, and we have found the population to grow (or shrink) exponentially. Figure 2 (a) shows the value of $a(n)$ starting with an initial population of 1000 birds, and for different values of the growth rate r. Depending on the sign of r, the population grows, remains constant ($r = 0$), or shrinks. These plots must be considered approximations of reality even if the basic assumptions listed earlier hold. This is because multiplication of $a(0)$ by powers of R will most of the time yield non-integer values, which make no sense for the size of a population.

Of course, r cannot be less than -1: the worst case (from the birds' point of view) is when every bird dies and none is born, $b = 0$ and $d = 1$, which corresponds to $r = 0 - 1 = -1$. On the other hand, the growth rate is in principle unbounded above, since there is no fixed limit on how many birds can be born in any one year. Of course, environmental factors impose limits, but a very large value of r is not meaningless, while a value smaller than -1 is.

On the other hand, equations (1) and (2) make at least *mathematical* sense for any value of r, and there are other physical scenarios where $r < -1$ makes sense even in practice. Think for instance of $a(n)$ representing money owed to an investor, and of r as a return on investment. In bad times, the loss could conceivably be greater than the amount invested, and $a(n)$ would then turn negative, meaning that the investor has a debt rather than a credit. If $a(n)$ is money (or a voltage, or some other quantity), then even non-integer values may make sense.

[7] See J. Cox, R. Kautz, M. MacLaughlin, and T. Gilbert, *Closing the Gaps in Florida's Wildlife Conservation System.* Office of Environmental Services, Florida Game and Fresh Water Fish Commission, Tallahassee, FL, 1994.



(a) (b)

Figure 2: (a) Size of the sand-hill crane population, starting with $a(0) = 1000$ birds, for different values of the growth rate r. (b) Plots of equation (2) for (physically impossible) growth rates smaller than -1.

Figure 2 (b) shows plots of equation (2) for values of r of -1.95, -2, and -2.05, which correspond to $R = -0.95, -1, -1.5$. While positive values of R (Figure 2 (a)) yield monotonic behaviors (increasing, constant, or decreasing depending on whether R is greater than, equal to, or less than 1), negative values of R (Figure 2 (b)) yield oscillatory patterns whose *envelope* increases, remains constant, or decays depending on whether the *magnitude* of R is greater than, equal to, or less than 1.

The oscillations in Figure 2 (b) are somewhat uninteresting, as the only oscillation that is possible with a system of this type is a switch of sign at every new time instant. We will see more interesting oscillations with slightly more complex systems.

2.2 DDDS Example 2: Hacking

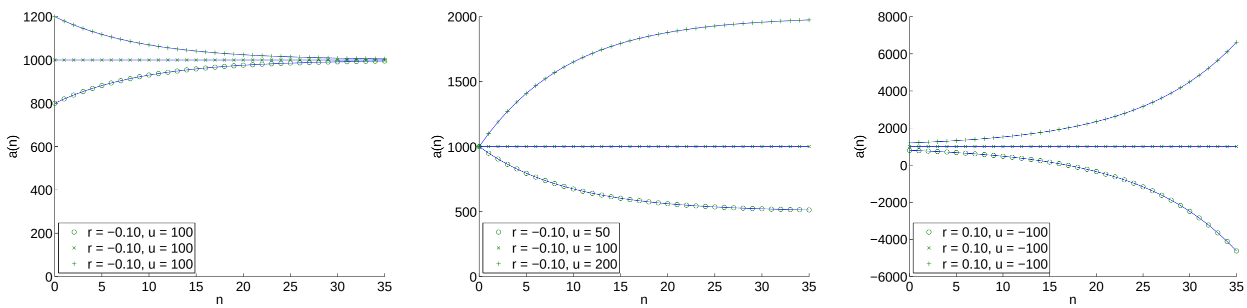
Without hacking, all a population can do (in the simple model developed earlier) is to grow indefinitely, remain constant, or shrink to zero. Hacking can prevent a population from becoming extinct, and is of course useful only for populations that would otherwise shrink, so that $r < 0$. Let us investigate what happens with a constant hacking policy, in which

$$u(n) = u \; .$$

In other words, we add a constant number u of chicks every year to a population that would otherwise shrink ($r < 0$). The reverse, population control, is also interesting: we eliminate u birds every year from a population that would otherwise grow indefinitely ($r > 0$).

Rather than *solving* equation (1) in these new situations (something we will do in a more general setting later), we merely ask the question of *asymptotics*: What happens to the population in the long run? In other words, will the population settle on a fixed size? After all, in the hacking case, the population itself would tend to become extinct, while we keep adding chicks to it, so it makes sense to expect that some sort of equilibrium might be reached. What happens in population control is somewhat different: If we cull too few animals every year, we eventually lose the battle against population explosion. However, if we cull enough animals, the population will shrink, and at some point the constant number of animals we cull every year will dominate over growth (which is proportional to population size), and the population will go to zero.

If an equilibrium situation occurs, it is called a *fixed point* in mathematics, or a *steady state* in engineering. Let us first try the empirical approach. Figure 3 (a) shows three plots obtained for the same value $r = -0.1$ and $u = 100$, starting with three different initial populations. Figure 3 (b) varies the hacking policy for a given starting point and growth rate: $a(0) = 1000$ and $r = -0.1$, with $u = 50, 100, 200$. Finally, Figure 3 (c) explores population control with $r = 0.1$ (a naturally growing population), and $u = -100$, and three different starting conditions, $a(0) = 800, 1000, 1200$.



(a) (b) (c)

Figure 3: Size of the sand-hill crane population under different conditions: (a) Same, negative growth rate r and hacking policy u, different starting populations $a(0)$. (b) Same, negative growth rate r and starting population $a(0)$, different hacking policy u. (c) Same, positive growth rate r and culling policy u, different starting populations $a(0)$.

These Figures let us make a few conjectures as to what might happen in general. First, Figure 3 (a) seems to indicate that the initial population has no bearing on the steady state population, since all three plots converge to the same value of 1000 animals. Figure 3 (b) shows that different hacking policies lead to different steady state populations, even when starting from the same initial number, and with the same (negative) growth rate. Figure 3 (c) supports the conjecture that there are only two ways to reach equilibrium in the population control scenario ($r > 0$): Either already start at the magic number (1000 in the figure), so that natural growth each year happens to match exactly the number of culled animals, or start at a lower number, and then the population becomes eventually extinct. The latter can hardly be called an "equilibrium:" If $a(n)$ were allowed to become negative (remember Figure 2 (b)), it seems that it would tend to negative infinity.

These conjectures are simple to verify mathematically. If there is a steady state a, that means that once we get there we stay there. In symbols, there must be a value n_0 such that

$$a(n) = a \quad \text{for} \quad n \geq n_0 \; .$$

Then, if we wait one more year, we can replace a for both $a(n)$ and $a(n+1)$ in equation (1), to obtain

$$a = (1+r)a + u$$

(recall that we chose to make $u(n)$ a constant as well). This equation can be solved for a to yield

$$a = -\frac{u}{r} \ . \tag{3}$$

So by assuming that a steady state exists, we found its value. This formula for a does not contain $a(0)$, and this proves our conjecture that the initial value has no bearing on the steady state, if one exists.

Please note carefully what the steady state means: *if* we get there, *then* we stay there forever. Figure 3 (c) shows that the steady state may never be reached. In that case, if we happen to hit it from the very beginning, that is, if

$$a(0) = a = -\frac{u}{r} = -\frac{-100}{0.1} = 1000 \ ,$$

then we remain there forever: every year, 100 animals die naturally, but we add exactly 100, so the population remains unaltered. However, this is a very brittle state of affairs. If we had started with $a(0) = 1001$ birds, the population would have exploded (try a few iterations of equation (1)). If we had started with $a(0) = 999$, the population would eventually be wiped away. This type of steady state is called *unstable*: as soon as we move away from it, we never go back. On the other hand, the steady states in Figure 3 (a) and, as it turns out, Figure 3 (b), are *stable*: even if for some reason we move away from it, we'll eventually revert to the same steady state over time. So in the population control case, the steady state exists, and is still independent of the initial population, because the value is given by equation (3). However, we may never reach that state, depending on the value of the initial population. More formally and generally:

Definition: A steady state a for the discrete, deterministic, dynamic system with equation

$$a(n+1) = f(a(n))$$

is said to be *stable* if there is a value $\delta > 0$ such that if $|a(n) - a| < \delta$ for some value of n then

$$\lim_{n\to\infty} a(n) = a \ .$$

In words, a steady-state value a for a system is stable if the system's state $a(n)$ tends eventually to a if it ever gets to a value that, although distinct from a, is close enough to it.

It should not be too hard to convince yourself that the general rule for stability for the sand-hill crane example is as follows: If $r < 0$, then the steady state of a system described by equation (1) is stable. If $r > 0$, then the steady state in question is unstable. When $r = 0$ (a situation of purely theoretical interest), the system is said to be *marginally stable*. In this case, equation (1) (with

$u(n) = u$) becomes

$$a(n+1) = a(n) + u$$

so the population changes every year exactly by the number of animals we add or remove. In this case, the ratio in equation (3) becomes undefined, and no steady state exists, except when $u = 0$ (a very static scenario indeed).

Rather than proving the stability condition above for the sand-hill crane situation, we state and prove it in a more general setting:

> Given any differentiable function $f : \mathbb{R} \to \mathbb{R}$, if a is a steady state value for the discrete, deterministic, dynamic system with equation
>
> $$a(n+1) = f(a(n)) , \tag{4}$$
>
> then a is stable if and only if
>
> $$|f'(a)| < 1 .$$

Here, the prime denotes differentiation.

Before we prove this, note that equation (1) with $u(n) = u$ meets the assumptions of this theorem, because the function $f(a)$ in this case is

$$f(a) = (1+r)a + u$$

and its derivative with respect to a is

$$f'(a) = 1 + r .$$

We have

$$|f'(a)| < 1$$

whenever

$$|1+r| < 1 \quad \text{that is,} \quad -1 < 1 + r < 1 .$$

This is equivalent to

$$-2 < r < 0 \quad \text{or} \quad -1 < R < 1$$

(recall that we defined $R = 1 + r$). In words, the steady state value (3) for equation (1) with $u(n) = u$ is stable if and only if the growth rate r is negative, but no lower than -2. This last qualification is irrelevant for populations, because $r < -1$ already leads to negative $a(n)$.

Begin optional part

Let us now prove the general result above. We can approximate equation (4) by using the first-order Taylor series expansion of f around the steady state a:

$$a(n+1) = f(a(n)) \approx f(a) + f'(a)\,(a(n) - a) .$$

Since a is a steady-state value, we have

$$a = f(a)$$

so that

$$a(n+1) \approx a + f'(a)\,(a(n) - a) \quad \text{that is,} \quad a(n+1) - a \approx f'(a)\,(a(n) - a)\ .$$

If we define the discrepancy between the actual state $a(n)$ and the steady state a as

$$d(n) = a(n) - a\ ,$$

the approximate equality above can be rewritten as follows:

$$d(n+1) \approx f'(a)\,d(n)\ .$$

We know from equation (2) that the exact version of this approximate equation has solution

$$d(n) = [f'(a)]^n\,d(0)$$

which is an exponential function of n. This function decays to zero (that is, $a(n)$ approaches a) if and only if the base $f'(a)$ of the exponential has magnitude smaller than one:

$$|f'(a)| < 1\ .$$

So we have proven the theorem, as long as we show that we can afford the approximation entailed by the truncated Taylor series expansion above. However, this approximation is legal, because stability only requires the state $a(n)$ to tend to the steady state value a when the system starts close enough to it.

End optional part

2.3 DDDS Example 3: Government Deficit Spending

The welfare of a nation can be quantified by its Gross Domestic Product (GDP), which is the total value of the goods and services that the nation produces in a given time period. The GDP can be measured directly, by tallying all the goods and services. However, it is practically easier to follow the money trail, that is, to account for all the expenditures made in the given economy. In theory, total output and total expenditures are the same, because money is spent on goods and services.[8]

According to the British economist John Maynard Keynes, and ignoring imports and exports[9] there are three parties with money in a national economy: the government, which can engage in *deficit spending*, that is, in spending more than its income; consumers, who buy goods and services; and private investors, that is, owners of companies who spend money to buy capital resources, employ personnel, and purchase raw materials or services to produce more goods and services.

Suppose that we measure expenditures at regular time intervals, say, every three months. In addition, we consider a *marginal* analysis, in the sense that we analyze the effects of government deficit spending superimposed to an economy that works on its own. In other words, we want to measure the additional effect that, say, an additional dollar in government spending has on GDP,

[8] In practice, discrepancies exist because of the time lag between inventory levels in different accounting practices.

[9] This assumption was more closely satisfied in the Thirties than it is today.

compared to what the GDP would be in the absence of this increment. Then, the additional GDP $a(n)$ in period n is the sum of the increments in the expenses by these three parties:

$$a(n) = c(n) + i(n) + u(n)$$

where $c(n)$ is the change of consumer spending in the same period, $i(n)$ is the change of investment, and $u(n)$ is the change in government deficit spending.

A classical model[10] developed by Hansen and Samuelson at the tail end of the Great Depression makes the following assumptions about consumption expenditure and private investment:

- Consumer spending in period n is proportional to the GDP in the previous period through a constant α called the *marginal propensity to consume*:

  $$c(n) = \alpha a(n-1) \; .$$

  This assumption relies on the observation that increased GDP entails increased income, which in turn buoys consumer confidence in the subsequent period. The constant α is nonnegative, but can also be greater than one because of credit purchases.[11]

- Private investment is proportional to the increase in consumer spending since the previous period:

  $$i(n) = \beta[c(n) - c(n-1)] \; ,$$

  since investors make new investments to meet the increased demand, in the hope of future profits. The constant β is called the *acceleration*, and is also nonnegative (and otherwise unrestricted).

These formulas can be compacted into one as follows:

$$\begin{aligned} a(n) &= c(n) + i(n) + u(n) \\ &= \alpha a(n-1) + \beta[c(n) - c(n-1)] + u(n) \\ &= \alpha a(n-1) + \beta[\alpha a(n-1) - \alpha a(n-2)] + u(n) \; . \end{aligned}$$

If we move time forward one step, for consistency with other formulas we have used and will be using, we can write

$$a(n+1) = \alpha a(n) + \beta[\alpha a(n) - \alpha a(n-1)] + u(n+1) \; .$$

If we let

$$b = \alpha \frac{1+\beta}{2} \quad \text{and} \quad c = \alpha\beta$$

[10] P. A. Samuelson, Interactions between the multiplier analysis and the principle of acceleration, *The Review of Economics and Statistics*, MIT Press, 21(2):75–78, 1939.

[11] The first bank credit card was issued by the Flatbush National Bank of Brooklyn, New York, in 1946. However, company credit cards had been in existence for decades before that.

we obtain

$$a(n+1) = 2ba(n) \; - \; ca(n-1) \; . \tag{5}$$

By construction, we have

$$b \geq 0 \quad \text{and} \quad c \geq 0 \; .$$

The division by two in the definition of b is of course arbitrary, but it will make some later notation a bit simpler.

Equation (5) seems only slightly different from equation (1), because of the presence of the additional term $-ca(n-1)$. This term comes from a double delay in the process: new investments are a function of consumer spending levels in the past *two* periods. This double delay makes equation (5) a *second-order* system, while equation (1) represents a *first order* system.

The additional delay makes a big difference in the qualitative behavior of the GDP $a(n)$ as a function of time n. Figure 4 shows plots of $a(n)$ for different values of α and β, and therefore b and c, under the simplifying assumption that the government spending increment is constant over time, and equal to 1 (say, one dollar).

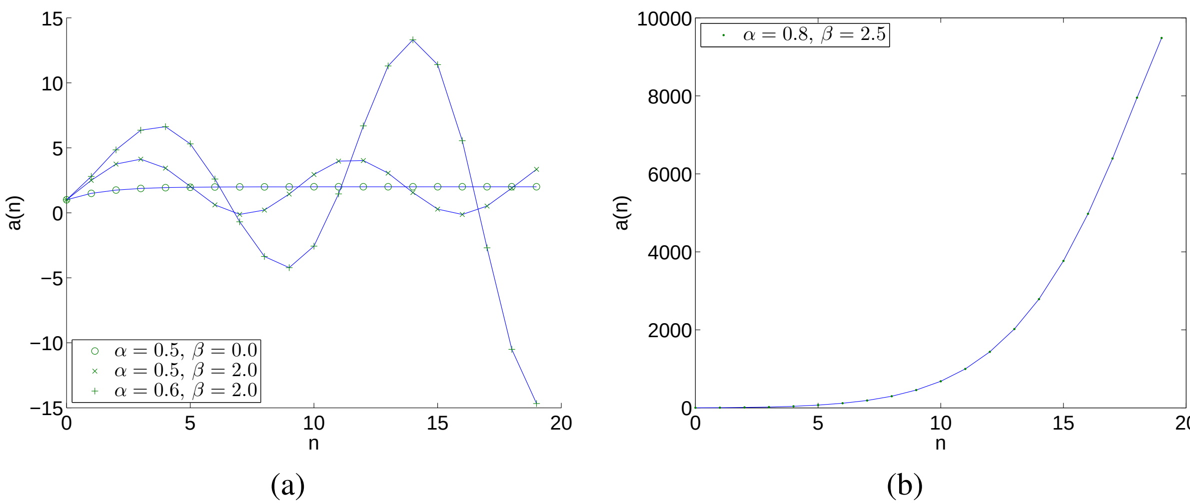


Figure 4: Plots of national income $a(n)$ versus accounting period number n for different values of the marginal propensity α to consume and investment acceleration β. The case $\alpha = 0.8, \beta = 2.5$ is plotted on a separate graph (b) because of the very different scale of the ordinate axis.

Note that negative values for $a(n)$ arise in Figure 4(a). This is consistent with the fact that we are performing a marginal analysis, as explained earlier. For instance, a negative investment value means that the investment level is lower than it would be in the absence of the given change in government spending.

The most striking feature of the two parts of Figure 4, taken together, is the diversity of behaviors with different parameter values. If there is no change in investment as a result of changes in demand, that is, if $\beta = 0$ (investors have no willingness to take risks), then the increment in GDP increases monotonically (plot with circles in panel (a) of the Figure), to approach a steady-state value that depends on the marginal propensity to consume, and is equal to two units in the example, with $\alpha = 0.5$: A single additional dollar of government spending causes a long-term increase of two dollars in the GDP, and this is the effect of accumulated, increased consumer spending over time.

Adding the accelerating effect of private investment ($\beta = 2$) to the same propensity to consume ($\alpha = 0.5$) yields the oscillating plot with the 'x' markers. This oscillation diminishes very slowly in amplitude, and reaches the same level of two units as for the previous case, although the plot in the Figure is truncated too early for this to be evident. Recall that oscillations never occurred in a first-order system, except possibly for trivial sign changes at every iteration.

Even a slight increase in propensity to consume has a staggering effect ($\alpha = 0.6$ and $\beta = 2$, graph with the '$+$' markers). Instead of dying down, the periodic fluctuations in the GDP increase in magnitude over time.

Finally, an additional increase in both propensity to spend and investment acceleration ($\alpha = 0.8$ and $\beta = 2.5$, Figure 4(b)) has an explosive effect, and the GDP increases exponentially. As long as both consumer confidence and investor optimism are high, a moderate but steady cash infusion from government spending has dramatic consequences on the welfare of a nation.

Of course, this model of the economy is overly simplistic, and other factors would soon intervene to curb exponential growth. Nonetheless, this simple scenario shows the great potential and dangers of economic policy. Also, the fact that the same model leads to very different outcomes depending on the values of the parameters indicates that economic policy decisions need to be based on quantitative models, rather than qualitative considerations.