3 Matrices and Vectors

This Section is a very concise introduction to the algebra of matrices and vectors. This will let us study mathematical models of the types we saw in the preceding illustrative examples in a more general, unified fashion.

3.1 Matrices

A (real) matrix of size $m \times n$ is an array of $mn$ real numbers arranged in $m$ rows and $n$ columns:

$$A = \begin{bmatrix}
a_{11} & \cdots & a_{1n} \\
\vdots & & \vdots \\
a_{m1} & \cdots & a_{mn}
\end{bmatrix}.$$  

The $n \times m$ matrix $A^T$ obtained by exchanging rows and columns of $A$ is called the transpose of $A$. A matrix $A$ is said to be symmetric if $A = A^T$.

The sum of two matrices of equal size is the matrix of the entry-by-entry sums, and the scalar product of a real number $a$ and an $m \times n$ matrix $A$ is the $m \times n$ matrix of all the entries of $A$, each multiplied by $a$. The difference of two matrices of equal size $A$ and $B$ is

$$A - B = A + (-1)B.$$  

The product of an $m \times p$ matrix $A$ and a $p \times n$ matrix $B$ is an $m \times n$ matrix $C$ with entries

$$c_{ij} = \sum_{k=1}^{p} a_{ik}b_{kj}.$$  

Examples

Let

$$A = \begin{bmatrix}3 & 0 & -2 \\1 & -1 & 0\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}2 & -1 \\1 & 3\end{bmatrix}.$$  

Then,

$$A^T = \begin{bmatrix}3 & 1 \\0 & -1 \\-2 & 0\end{bmatrix}, \quad B^T = \begin{bmatrix}2 & 1 \\-1 & 3\end{bmatrix}$$  

and

$$C = BA = \begin{bmatrix}2 \cdot 3 + (-1) \cdot 1 & 2 \cdot 0 + (-1) \cdot (-1) & 2 \cdot (-2) + (-1) \cdot 0 \\1 \cdot 3 + 3 \cdot 1 & 1 \cdot 0 + 3 \cdot (-1) & 1 \cdot (-2) + 3 \cdot 0\end{bmatrix} = \begin{bmatrix}5 & 1 & -4 \\6 & -3 & -2\end{bmatrix}.$$  

The product $AB$ is not defined, because $A$ and $B$ have incompatible sizes. Furthermore,

$$3A = \begin{bmatrix}3 \cdot 3 & 3 \cdot 0 & 3 \cdot (-2) \\3 \cdot 1 & 3 \cdot (-1) & 3 \cdot 0\end{bmatrix} = \begin{bmatrix}9 & 0 & -6 \\3 & -3 & 0\end{bmatrix}.$$
\[ A + C = \begin{bmatrix} 3 & 0 & -2 \\ 1 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 5 & 1 & -4 \\ 6 & -3 & -2 \end{bmatrix} = \begin{bmatrix} 3 + 5 & 0 + 1 & -2 + (-4) \\ 1 + 6 & -1 + (-3) & 0 + (-2) \end{bmatrix} = \begin{bmatrix} 8 & 1 & -6 \\ 7 & -4 & -2 \end{bmatrix} \]

and

\[ A - C = \begin{bmatrix} 3 & 0 & -2 \\ 1 & -1 & 0 \end{bmatrix} - \begin{bmatrix} 5 & 1 & -4 \\ 6 & -3 & -2 \end{bmatrix} = \begin{bmatrix} 3 - 5 & 0 - 1 & -2 - (-4) \\ 1 - 6 & -1 - (-3) & 0 - (-2) \end{bmatrix} = \begin{bmatrix} -2 & -1 & 2 \\ -5 & 2 & 2 \end{bmatrix}. \]

### 3.2 Vectors

A (real) \( n \)-dimensional vector is an \( n \)-tuple of real numbers

\[ \mathbf{v} = (v_1, \ldots, v_n). \]

There is a natural, one-to-one correspondence between \( n \)-dimensional vectors and \( n \times 1 \) matrices:

\[ (v_1, \ldots, v_n) \leftrightarrow \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}. \]

The matrix on the right is called the column vector corresponding to the vector on the left.

There is also a natural, one-to-one correspondence between \( n \)-dimensional vectors and \( 1 \times n \) matrices:

\[ (v_1, \ldots, v_n) \leftrightarrow \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}. \]

The matrix on the right is called the row vector corresponding to the vector on the left.

If \( \mathbf{a} \) is a vector, then the symbol \( \mathbf{a} \) also denotes the corresponding column vector, so that the corresponding row vector is \( \mathbf{a}^T \).

All algebraic operations on vectors are inherited from the corresponding matrix operations, when defined. In addition, the inner product of two \( n \)-dimensional vectors

\[ \mathbf{a} = (a_1, \ldots, a_n) \quad \text{and} \quad \mathbf{b} = (b_1, \ldots, b_n) \]

is the real number equal to the matrix product \( \mathbf{a}^T \mathbf{b} \). It is easy to verify that this is also equal to \( \mathbf{b}^T \mathbf{a} \). Two vectors that have a zero inner product are said to be orthogonal.

The norm of a vector \( \mathbf{a} \) is

\[ \| \mathbf{a} \| = \sqrt{\mathbf{a}^T \mathbf{a}}, \]

obviously a nonnegative number. A unit vector is a vector with norm one.

### Examples

The vector \( \mathbf{a} = (2, -1, 0) \) corresponds to row vector \( \mathbf{a}^T = [2, -1, 0] \) and to column vector

\[ \mathbf{a} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}. \]
The inner product of \(a\) and \(b = (1, 0, -1)\) is

\[
a^T b = [2, -1, 0] \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 2 \cdot 1 + (-1) \cdot 0 + 0 \cdot (-1) = 2,
\]

and the norm of \(a\) is

\[
\|a\| = \sqrt{a^T a} = \sqrt{2 \cdot 2 + (-1) \cdot (-1) + 0 \cdot 0} = \sqrt{5} \approx 2.2361.
\]

The vector

\[
c = \frac{1}{\sqrt{5}} a = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} \\ 0 \end{bmatrix} \approx \begin{bmatrix} 0.8944 \\ -0.4472 \\ 0 \end{bmatrix}
\]

has unit norm:

\[
\|c\| = \sqrt{c^T c} = \sqrt{\left(\frac{2}{\sqrt{5}}\right)^2 + \left(\frac{-1}{\sqrt{5}}\right)^2 + \left(\frac{0}{\sqrt{5}}\right)^2} = \sqrt{\frac{2^2 + (-1)^2 + 0^2}{5}} = \sqrt{\frac{5}{5}} = \sqrt{1} = 1.
\]