Ch. 7 - Pushdown Automata

A DFA = \((Q, \Sigma, \delta, q_0, F)\)

Modify DFA by adding a stack. New machine is called Pushdown Automata (PDA).

Definition: Nondeterministic PDA (NPDA) is defined by

\[ M = (Q, \Sigma, \Gamma, \delta, q_0, z, F) \]
where

- \( Q \) is a finite set of states
- \( \Sigma \) is the tape (input) alphabet
- \( \Gamma \) is the stack alphabet
- \( q_0 \) is the initial state
- \( z \) is the start stack symbol, (bottom of stack marker), \( z \in \Gamma \)
- \( F \subseteq Q \) is the set of final states.
- \( \delta : Q \times (\Sigma \cup \{\lambda\}) \times \Gamma \rightarrow \text{finite subsets of } Q \times \Gamma^* \)

**Example of transitions**

\[
\delta(q_1,a,b) = \{(q_3,b),(q_4,ab),(q_6,\lambda)\}
\]

Meaning: If in state \( q_1 \) with “a” the current tape symbol and “b” the symbol on top of the stack, then pop “b”, and either

- move to \( q_3 \) and push “b” on stack
- move to \( q_4 \) and push “ab” on stack (“a” on top)
- move to \( q_6 \)

Transitions can be represented using a transition diagram.

The diagram for the above transitions is:

Each arc is labeled by a triple: \( x,y,z \) where \( x \) is the current input symbol, \( y \) is the top of stack symbol which is popped from the stack, and \( z \) is a string that is pushed onto the stack.

**Instantaneous Description:**

\[(q,w,u)\]

Notation to describe the current state of the machine \( q \), unread portion of the input string \( w \), and the current contents of the stack \( u \).
Description of a Move:

\[(q_1, aw, bx) \vdash (q_2, w, yx)\]

iff

**Definition** Let \( M = (Q, \Sigma, \Gamma, \delta, q_0, z, F) \) be a NPDA. \( L(M) = \{ w \in \Sigma^* \mid (q_0, w, z) \xrightarrow{*} (p, \lambda, u), p \in F, u \in \Gamma^* \} \). The NPDA accepts all strings that start in \( q_0 \) and end in a final state.

**Example:** \( L = \{ a^n b^n \mid n \geq 0 \} \), \( \Sigma = \{ a, b \} \), \( \Gamma = \{ z, a \} \)

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**Another Definition for Language Acceptance**

NPDA \( M \) accepts \( L(M) \) by empty stack:

\[ L(M) = \{ w \in \Sigma^* \mid (q_0, w, z) \xrightarrow{*} (p, \lambda, \lambda) \} \]
Example: $L=\{a^nb^mc^{n+m}|n, m > 0\}$, $\Sigma = \{a, b, c\}$, $\Gamma = \{0, z\}$

Example: $L=\{ww^R|w \in \Sigma^+\}$, $\Sigma = \{a, b\}$, $\Gamma = \{z, a, b\}$

Example: $L=\{ww|w \in \Sigma^*\}$, $\Sigma = \{a, b\}$

Examples for you to try on your own: (solutions are at the end of the handout).

- $L=\{a^nb^m|m > n, m, n > 0\}$, $\Sigma = \{a, b\}$, $\Gamma = \{z, a\}$
- $L=\{a^nb^{n+m}c^m|n, m > 0\}$, $\Sigma = \{a, b, c\}$
- $L=\{a^n b^{2n}|n > 0\}$, $\Sigma = \{a, b\}$
**Definition:** A PDA $M=(Q,\Sigma,\Gamma,\delta,q_0,z,F)$ is deterministic if for every $q \in Q$, $a \in \Sigma \cup \{\lambda\}$, $b \in \Gamma$

1. $\delta(q,a,b)$ contains at most 1 element
2. if $\delta(q,\lambda,b) \neq \emptyset$ then $\delta(q,c,b) = \emptyset$ for all $c \in \Sigma$

**Definition:** $L$ is DCFL iff $\exists$ DPDA $M$ s.t. $L=L(M)$.

**Examples:**

1. Previous pda for $\{a^nb^n|n \geq 0\}$ is deterministic?
2. Previous pda for $\{a^n b^m c^{n+m}|n,m > 0\}$ is deterministic?
3. Previous pda for $\{ww^R|w \in \Sigma^+\}, \Sigma = \{a,b\}$ is deterministic?
Example: \( L = \{ a^n b^m \mid m > n, m, n > 0 \} \), \( \Sigma = \{ a, b \} \), \( \Gamma = \{ z, a \} \)

Example: \( L = \{ a^n b^{n+m} c^m \mid n, m > 0 \} \), \( \Sigma = \{ a, b, c \} \)

Example: \( L = \{ a^n b^{2n} \mid n > 0 \} \), \( \Sigma = \{ a, b \} \)