1 The Gram-Schmidt orthogonalization process

Provided with a set of linearly indepedent vectors v_j, $j = 0, 2, \cdots$, in an inner product space V with the inner product

$$\langle u, v \rangle, \quad u, v \in V,$$

the Gram-Schmidt process generates routinely a set of orthogonal vectors q_k, with respect to the inner product $\langle \cdot, \cdot \rangle$, such that

$$\begin{aligned} \langle q_i, q_j \rangle &= c_i^2 \, \delta_{ij}, \\ \text{span}\{v_1, \cdots, v_k\} &= \text{span}\{q_1, \cdots, q_k\}, \quad k = 1, 2, \cdots \end{aligned} \tag{1}$$

where δ_{ij} denotes the Diract delta function, and c_i is the 2-norm of q_i induced by the inner product. The orthogonalization procedure is as simple as follows,

$$\begin{aligned} q_0(x) &= v_0(x), \\ q_{k+1}(x) &= v_{k+1}(x) - \sum_{j=0}^{k} q_j \frac{\langle q_j(x), v_{k+1} \rangle}{\langle q_j, q_j \rangle}. \end{aligned} \tag{2}$$

At step $k + 1$, the components of q_j in v_{k+1}, $j \le k$, are eliminated by projections.

Note this yields a QR factorization.

2 GS orthogonalization of power functions

Consider the space of univariate polynomials on an interval. Let $\langle \cdot, \cdot \rangle$ be a well-defined inner product,

$$\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)w(x)dx. \tag{3}$$

One obtains a family of orthogonal polynomials by applying the GS procedure on the power functions in increasing powers,

$$1, x, x^2, x^3, \cdots, x^k, \cdots.$$

For finite closed intervalus, one may consider the interval $[-1, 1]$ only, by translation and scaling in the indepdent variable. The well known orthogonal polynimail families include

- Legendre : $w(x) = 1$

- Chebyshev (I) : $w(x) = \dfrac{1}{\sqrt{1-x^2}}$
- Chebyshev (II) : $w(x) = \sqrt{1-x^2}$
- Jacobi : $w(x) = (1+x)^\alpha (1-x)^\beta$

For semi-infinite intervals, we have the following examples on $[0, \infty)$.

- Laguerre : $w(x) = e^{-x}$
- Generalized Laguerre : $w(x) = x^k e^{-x}$

The weight function must decay and vanishsing as x goes to infinity (why?). On $(-\infty, \infty)$, we have the Hermite polynomials with the weigh function $w(x) = e^{-x^2}$.

In each of these orthogonal polynimal families, there is a three-term recurrsion. To make this easy to see, we may replace $v_{k+1} = xx^k$ by $xq_k(x)$ at step $k+1$ in (2),

$$q_{k+1}(x) = xq_k(x) - \sum_{j=0}^{k} q_j(x) \frac{\langle q_j, xq_k \rangle}{\langle q_j, q_j \rangle}.$$

Notice that

$$\langle q_j, xq_k \rangle = \langle xq_j, q_k \rangle = 0, \quad (j+1) < k.$$

We therefore have

$$\begin{aligned} q_{k+1} &= (x - \alpha_k)\, q_k(x) - \beta_k\, q_{k-1}(x), \\ \alpha_k &= \frac{\langle q_k, xq_k \rangle}{\langle q_k, q_k \rangle}, \quad \beta_k = \frac{\langle xq_{k-1}, q_k \rangle}{\langle q_{k-1}, q_{k-1} \rangle}. \end{aligned} \tag{4}$$