1. Consider two graphs $G_A = (V, E_A)$ and $G_b = (V, E_b)$ with the same vertex set and $E_A \subset E_b$. Describe the relationships among the smallest non-zero and largest eigenvalues between the corresponding Laplacian matrices.

   Optional. The relationships among the interior eigenvalues between the two Laplacian matrices.

2. Let L be the Laplacian matrix of a graph. Let λ_2 be the Fiedler's eigenvalue. Prove or disprove that

$$\lambda_2^{-1} = \max_{sum x=0, x\neq 0} \frac{x^{\mathrm{T}} A^k x}{x^{\mathrm{T}} A^{k+1} x}$$

   for any positive integer k. Give an upper bound on λ_2.

3. Describe a division-free or inversion-free procedure for extracting the positive root of a symmetric positive definite matrix. Specify the condition(s).

4. Describe a fast approach to compute the eigenvalues of a circulant matrix.

5. Let x_k be an iterative result for the solution of the system of linear equations $Ax = b$. Construct an perturbed error matrix E_k, based on computed quantities such that

$$(A + E_k)x_k = b.$$

6. Experiment in matlab with the following particular difference equation

$$\begin{aligned} x_{j+1} = x_j + h(w_0 x_j + w_1 x_{j-1}) &= 2.25x_j - 0.5x_{j-1}, \\ x(0) = 1/3, \quad x(1) &= 1/12. \end{aligned}$$

   By analysis, the sequence is monotonically decreasing (which can be easily verified). The numerical sequence behaves differently at and from certain point. Give an analytical explanation.

7. Assume that A is nonsingular. Verify that A^{-1} is a polynomial in A.

8. Consider the following procedure for a nonsingular matrix A that has a complete eigen-vector system. Consider the following iteration procedure, provided with $u_0 \neq 0$,

$$v_k := A * u_k; \quad u_k := v_k / \|v_k\|_\infty.$$

Make comments on whether or not the vector sequence convergence, if yes, make further comments on the convergence rate and and the property of the vector at the limit.

9. Describe two-levels Jacobi iterations with finite-step termination for solving a large and sparse system of linear equations.

   Optional. Describe the error propagation matrix at the outer level.

10. Assume that A is symmetric positive definite. Consider the vector sequence $\{b, Ab, A^b, \cdots, A^k b, \cdots\}$.

    (a) Verify there is a finite upper bound on the number of linearly independent vectors in the sequence. Provide an upper bound.

    (b) Verify that
    $$\langle u, v\rangle_A = u^{\mathrm{T}} A v$$
    is a well defined inner product.

    (c) Orthogonalize a leading subsequence of the vectors with respect to the inner product $\langle u, v\rangle_A$

    (d) Find recursive relationship among the orthogonalized vectors.