For review and self-study on extension from class discussion; no due date.

1. Describe a block Jacobi iteration for the 3D discrete Laplacian equations as in the last homework and describe a comparison to the scalar version, either by observation of experimental results or by analysis.

2. Understanding of the Conjugate Gradient (CG) method.

   Consider the solution of $Ax = b$, where A is an $n \times n$ symmetric and positive definite matrix and b is nonzero.

   (a) Describe a procedure for obtaining A-conjugate vectors from the Krylov vector sequence $\{b, Ab, A^2b, \cdots\}$.

   Describe the maximum number of such A-conjugate vectors with any specific pair of A and b.

   Provide a three-term recursion for efficient computation of the A-conjugate vectors, and describe the essential condition(s) that make the efficient recursion possible, in comparison to the general Gram-Schmidt orthogonalization procedure in terms of arithmetic operation account.

   (b) Make use of the three-term orthogonalization procedure for successive LS approximations to the solution, and hence make a connection to the CG method.

   (c) Assume that there are $k < n$ clusters in the eigenvalues of A. Verify or disapprove this statement : in exact arithmetic operation, the CG method converges in k steps.

   (d) **Optional**. The same set of A conjugate vectors can be used to 'tridiagonalize' the matrix A in the sense that

   $$A(p_1, p_2, \cdots, p_m) = (p_1, p_2, \cdots, p_m)T$$

   where T is a matrix of tridiagonal form and $m \leq n$.

3. Make some suggestion, with some good argument, for using both the CG method and a block Jacobi method.

4. Try the provided example code `tryWavelets.m` for image compression with discrete wavelets and try to find a way to reduce the effective number of the coefficients.

5. Quadrature design for numerical integration.

   (a) Polynomial based quadrature design.
   Provided with two quadratures

   $$Q_a(f) = \sum_{i=1}^{n_a} f(x_{a,i})\, w_{a,i}, \quad Q_b(f) = \sum_{j=1}^{n_b} f(x_{b,j})\, w_{b,j},$$

   that have the same precision degree d.

   • Compose a new quadrature with higher degree by a linear combination of the two quadratures. Specify the nodes and weights of the new quadrature.

   • Apply the scheme to the midpoint quadrature and the trapezoidal quadrature.

   (b) Consider the family of trigonometric polynomials in the exponential representation for convenience.
   Verify that for any fixed degree, the weights are equal. Pay attention to the structure of the matrix to determine the weights.

6. Review of miterm-exam problems.

Optional

Provide an algorithm for the best low-rank symmetric semi-definite data matrix recovery in the F-norm or 2-norm. Note that under the assumed conditions, the matrix norms correspond to certain vector norms on the vector of eigenvalues/singular values.

Make comments on the additional complexity in algorithm design if the matrix norm is changed to the so called nuclear norm.