Example linear program

- We make reproductions of two paintings

\[
\begin{align*}
\text{maximize} & \quad 3x + 2y \\
\text{subject to} & \quad 4x + 2y \leq 16 \\
& \quad x + 2y \leq 8 \\
& \quad x + y \leq 5 \\
& \quad x \geq 0 \\
& \quad y \geq 0
\end{align*}
\]

- Painting 1 sells for $30, painting 2 sells for $20
- Painting 1 requires 4 units of blue, 1 green, 1 red
- Painting 2 requires 2 blue, 2 green, 1 red
- We have 16 units blue, 8 green, 5 red
Solving the linear program graphically

\[ \text{maximize } 3x + 2y \]
\[ \text{subject to } \]
\[ 4x + 2y \leq 16 \]
\[ x + 2y \leq 8 \]
\[ x + y \leq 5 \]
\[ x \geq 0 \]
\[ y \geq 0 \]

optimal solution: 
\[ x=3, y=2 \]
Proving optimality

\[
\text{maximize } 3x + 2y \\
\text{subject to } \\
4x + 2y \leq 16 \\
x + 2y \leq 8 \\
x + y \leq 5 \\
x \geq 0 \\
y \geq 0
\]

Recall: optimal solution: 
\( x=3, y=2 \) 
Solution value = 9+4 = 13

How do we prove this is optimal (without the picture)?
Proving optimality…

\[
\begin{align*}
\text{maximize} & \quad 3x + 2y \\
\text{subject to} & \quad 4x + 2y \leq 16 \\
& \quad x + 2y \leq 8 \\
& \quad x + y \leq 5 \\
& \quad x \geq 0 \\
& \quad y \geq 0
\end{align*}
\]

We can rewrite the blue constraint as
\[
2x + y \leq 8
\]

If we add the red constraint
\[
x + y \leq 5
\]
we get
\[
3x + 2y \leq 13
\]

Matching upper bound!

(Really, we added .5 times the blue constraint to 1 times the red constraint)
Linear combinations of constraints

$maximize \ 3x + 2y$

$subject \ to$

$4x + 2y \leq 16$

$x + 2y \leq 8$

$x + y \leq 5$

$x \geq 0$

$y \geq 0$

$b(4x + 2y \leq 16) + g(x + 2y \leq 8) + r(x + y \leq 5) =$

$(4b + g + r)x + (2b + 2g + r)y \leq 16b + 8g + 5r$

$4b + g + r \text{ must be at least } 3$

$2b + 2g + r \text{ must be at least } 2$

Given this, minimize $16b + 8g + 5r$
Using LP for getting the best upper bound on an LP

\[
\begin{align*}
\text{maximize} & \quad 3x + 2y \\
\text{subject to} & \quad 4x + 2y \leq 16 \\
& \quad x + 2y \leq 8 \\
& \quad x + y \leq 5 \\
& \quad x \geq 0 \\
& \quad y \geq 0
\end{align*}
\]

\[
\begin{align*}
\text{minimize} & \quad 16b + 8g + 5r \\
\text{subject to} & \quad 4b + g + r \geq 3 \\
& \quad 2b + 2g + r \geq 2 \\
& \quad b \geq 0 \\
& \quad g \geq 0 \\
& \quad r \geq 0
\end{align*}
\]

the dual of the original program

- Duality theorem: any linear program has the same optimal value as its dual!
Modified LP

\[ \text{maximize } 3x + 2y \]

\[ \text{subject to} \]
\[ 4x + 2y \leq 15 \]
\[ x + 2y \leq 8 \]
\[ x + y \leq 5 \]
\[ x \geq 0 \]
\[ y \geq 0 \]

Optimal solution: \( x = 2.5, y = 2.5 \)
Solution value = \( 7.5 + 5 = 12.5 \)

Half paintings?
Integer (linear) program

\[
\begin{align*}
\text{maximize} & \quad 3x + 2y \\
\text{subject to} & \quad 4x + 2y \leq 15 \\
& \quad x + 2y \leq 8 \\
& \quad x + y \leq 5 \\
& \quad x \geq 0, \text{ integer} \\
& \quad y \geq 0, \text{ integer}
\end{align*}
\]

optimal IP solution: \(x=2, y=3\) (objective 12)

optimal LP solution: \(x=2.5, y=2.5\) (objective 12.5)
Mixed integer (linear) program

maximize $3x + 2y$

subject to

$4x + 2y \leq 15$

$x + 2y \leq 8$

$x + y \leq 5$

$x \geq 0$

$y \geq 0$, integer

optimal IP solution: $x=2$, $y=3$ (objective 12)

optimal LP solution: $x=2.5$, $y=2.5$ (objective 12.5)

optimal MIP solution: $x=2.75$, $y=2$ (objective 12.25)
Exercise in modeling: knapsack-type problem

• We arrive in a room full of precious objects
• Can carry only 30kg out of the room
• Can carry only 20 liters out of the room
• Want to maximize our total value
• Unit of object A: 16kg, 3 liters, sells for $11
  – There are 3 units available
• Unit of object B: 4kg, 4 liters, sells for $4
  – There are 4 units available
• Unit of object C: 6kg, 3 liters, sells for $9
  – Only 1 unit available
• What should we take?
Exercise in modeling: cell phones (set cover)

- We want to have a working phone in every continent (besides Antarctica)
- ... but we want to have as few phones as possible
- Phone A works in NA, SA, Af
- Phone B works in E, Af, As
- Phone C works in NA, Au, E
- Phone D works in SA, As, E
- Phone E works in Af, As, Au
- Phone F works in NA, E
Exercise in modeling: hot-dog stands

- We have two hot-dog stands to be placed in somewhere along the beach
- We know where the people that like hot dogs are, how far they are willing to walk
- Where do we put our stands to maximize #hot dogs sold? (price is fixed)